UNM-PNM 2022-2023 Math Contest Round 2 Solutions

1. Given positive real numbers a, b, c and d such that $\frac{a}{b} < \frac{c}{d} < 1$, arrange the following five numbers in order from least to greatest: $\frac{b}{a}$, $\frac{d}{c}$, $\frac{bd}{ac}$, $\frac{b+d}{a+c}$, 1.

   Answer: $1 < \frac{d}{c} < \frac{b+d}{a+c} < \frac{b}{a} < \frac{bd}{ac}$.

   Solution: Multiplying the given inequality $\frac{a}{b} < \frac{c}{d} < 1$ by $\frac{bd}{ac}$ yields $\frac{d}{c} < \frac{b}{a} < \frac{bd}{ac}$. Multiplying the given inequality $\frac{c}{d} < 1$ by $\frac{d}{c}$ yields $1 < \frac{d}{c}$. Combining these two results, we have $1 < \frac{d}{c} < \frac{b}{a} < \frac{bd}{ac}$. Now, $\frac{d}{c} < \frac{b}{a}$ implies $ad < bc$. Thus $ad + cd < bc + cd$, which implies $d(a + c) < c(b + d)$, hence $\frac{d}{c} < \frac{b+d}{a+c}$. Similarly, $ad+ab < bc+ab$ yields $\frac{b+d}{a+c} < \frac{b}{a}$. These results determine the following ordering: $1 < \frac{d}{c} < \frac{b+d}{a+c} < \frac{b}{a} < \frac{bd}{ac}$.

2. When added together, the perimeters of an equilateral triangle and a square have a total length of L. Find the side length of the triangle (in terms of L) that minimizes the sum of the areas of the triangle and square.

   Answer: $\frac{3L}{4\sqrt{3}+9}$, or equivalently, $\frac{(9-4\sqrt{3})L}{11}$.

   Solution: Let t be the side length of the triangle and s be the side length of the square. Summing the perimeters yields $3t + 4s = L$, thus $s = \frac{L-3t}{4}$. The area of the triangle is $\frac{1}{2}$(base)(height), where the base is t and the height is $\frac{\sqrt{3}}{2}t$ (this can be seen by dividing the equilateral triangle into two equivalent right triangles and applying the Pythagorean theorem to one of them). So the equilateral triangle has area $\frac{\sqrt{3}}{4}t^2$ and the square has area $s^2 = \frac{(L-3t)^2}{16}$. Summing these gives the total area of the triangle and square as a function of t:

$$A(t) = \frac{\sqrt{3}}{4}t^2 + \frac{(L-3t)^2}{16}.$$

   Expanding and simplifying yields the quadratic function

$$A(t) = at^2 + bt + c,$$

   where $a = \frac{9+4\sqrt{3}}{16}$, $b = -\frac{6L}{16}$, and $c = \frac{L^2}{16}$. The graph of $A(t)$ is a concave up parabola, so $A(t)$ attains a minimum value at the vertex, where

$$t = -\frac{b}{2a} = \frac{3L}{9+4\sqrt{3}} = \frac{(9-4\sqrt{3})L}{11}.$$

3. Suppose a, b, and c are positive real numbers such that $a < b < c < a + b$. Find the area (in terms of a, b, and c) of the 5-sided polygon bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = b$, and $x + y = c$.

   Answer: $ab - \frac{1}{2}(a + b - c)^2$

   Solution: The line $x + y = c$ intersects the horizontal line $y = b$ at the point $(c - b, b)$ and intersects the vertical line $x = a$ at the point $(a, c - a)$. We seek the area A of the 5-sided polygon with vertices $(0, 0)$, $(a, 0)$, $(a, c - a)$, $(c - b, b)$, $(0, b)$. This can be obtained by subtracting the area T of the triangle with vertices (a, b), $(c - b, b)$, $(a, c - a)$ from the area R of the rectangle with vertices $(0, 0)$, $(a, 0)$, (a, b), $(0, b)$. Thus

$$\begin{aligned} A &= R - T \\ &= ab - \frac{1}{2}(\text{base})(\text{height}) \\ &= ab - \frac{1}{2}(a - (c - b))(b - (c - a)) \\ &= ab - \frac{1}{2}(a + b - c)^2. \end{aligned}$$

4. Two players alternate shooting free throws in basketball. For each attempt, player 1 has a 1/3 probability of success and player 2 has a 1/4 probability of success. What is the probability that player 1 succeeds before player 2?

Answer: $\frac{2}{3}$

Solution: Let E be the event of interest: player 1 succeeds before player 2. Let A_j be the event that player 1 succeeds first on the jth shot. Note that $A_1, A_2, A_3, \ldots$ are disjoint sets and $E = \bigcup_{j=1}^{\infty} A_j$. Thus the probability of event E is given by

$$P(E) = P(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} P(A_j).$$

Player 1 has a $\frac{1}{3}$ probability of succeeding on his first shot, so $P(A_1) = \frac{1}{3}$. For A_2 to occur, player 1 and player 2 must both miss their 1st shot and player 1 must make his 2nd shot. Thus $P(A_2) = (\frac{2}{3})(\frac{3}{4})(\frac{1}{3})$. More generally, for A_j to occur, player 1 and player 2 must both miss their 1st $j-1$ shots, and player 1 must make his jth shot. Thus

$$P(A_j) = \left(\frac{2}{3}\right)^{j-1} \left(\frac{3}{4}\right)^{j-1} \left(\frac{1}{3}\right) = \frac{1}{3}\left(\frac{1}{2}\right)^{j-1}.$$

This yields

$$P(E) = \sum_{j=1}^{\infty} P(A_j) = \sum_{j=1}^{\infty} \frac{1}{3}\left(\frac{1}{2}\right)^{j-1} = \frac{1}{3}\left(\frac{1}{1-\frac{1}{2}}\right) = \frac{2}{3},$$

where we have used the geometric series $1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$ when $|r| < 1$.

Assuming the series converges, this result can also be obtained by writing

$$\begin{aligned} P(E) &= \frac{1}{3}\left(1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right) \\ &= \frac{1}{3} + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right) \\ &= \frac{1}{3} + \frac{1}{2}P(E), \end{aligned}$$

and then solving for $P(E)$.

The same equation for $P(E)$ can be obtained by using conditional probability. Let $P(A|B)$ denote the probability of event A given that event B has occurred. This is defined as $P(A|B) = \frac{P(A\cap B)}{P(B)}$. Let A_1^c denote the complement of A_1. Noting that $E \cap A_1 = A_1$ since $A_1 \subset E$, we have

$$\begin{aligned} P(E) &= P(E \cap (A_1 \cup A_1^c)) \\ &= P((E \cap A_1) \cup (E \cap A_1^c)) \\ &= P(A_1 \cup (E \cap A_1^c)) \\ &= P(A_1) + P(E \cap A_1^c) \\ &= P(A_1) + P(A_1^c|E)P(E). \end{aligned}$$

Given that event E occurs, A_1 will not occur if and only if player 1 and player 2 both miss their 1st shot. Thus

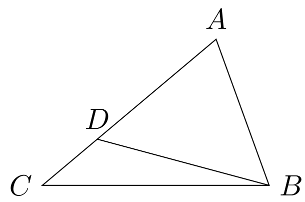
$$P(A_1^c|E) = \left(\frac{2}{3}\right)\left(\frac{3}{4}\right) = \frac{1}{2}.$$

Recalling that $P(A_1) = \frac{1}{3}$, we obtain the same equation for $P(E)$ that we saw before:

$$P(E) = \frac{1}{3} + \frac{1}{2}P(E),$$

whose solution is $P(E) = \frac{2}{3}$.

5. A triangle has vertices A, B, and C. Suppose point D is on the line segment $\overline{AC}$ such that $AB = AD$. If $\angle ABC - \angle ACB = 30°$, find $\angle CBD$.



Answer: $15°$

Solution: Let $x = \angle CBD$, $y = \angle ABD$ and $z = \angle ACB$. Then $\angle ABC = x + y$, and we are given that

$$x + y - z = 30°. \tag{1}$$

Triangle ABD is isosceles, and $AB = AD$ implies $\angle ABD = \angle BDA = y$. Since $\angle BDA$ and $\angle BDC$ are supplementary, we must have $\angle BDC = 180° - y$. Summing the angles in triangle BDC then yields

$$x + (180° - y) + z = 180°,$$

or equivalently

$$x - y + z = 0°. \tag{2}$$

Adding equations (1) and (2) yields $2x = 30°$, hence $x = 15°$.

6. Find a positive real number x such that $2[x] + [1 - x] = \dfrac{19}{x}$, where $[x]$ denotes the greatest integer less than or equal to x.

Answer: $\frac{19}{4} = 4.75$

Solution: We consider 2 cases: either x is an integer or it is not.

First, suppose x is an integer. Then $[x] = x$ and $[1 - x] = 1 - x$, so x must satisfy the equation

$$2x + 1 - x = \frac{19}{x}$$

which implies

$$x^2 + x - 19 = 0.$$

The quadratic formula gives $x = \frac{-1 \pm \sqrt{77}}{2}$, which shows that x cannot be an integer.

Now suppose x is a positive number that is not an integer. Then we can write $x = n + h$ where n is a nonnegative integer and $0 < h < 1$. This implies

$$[x] = [n + h] = n$$

and

$$[1 - x] = [1 - n - h] = [-n + (1 - h)] = -n$$

since $0 < 1 - h < 1$. Our equation then becomes

$$2n + -n = \frac{19}{n + h},$$

which can be rewritten as

$$n(n + h) = 19.$$

We can determine the integer n from the following estimates:

$$n^2 < n(n + h) = 19$$

and

$$(n+1)^2 > n(n+h) = 19,$$

which together yield

$$3 < \sqrt{19} - 1 < n < \sqrt{19} < 5.$$

The only integer n that satisfies the above condition is $n = 4$. We can then determine h by solving

$$4(4+h) = 19$$

which yields $h = \frac{19}{4} - 4 = \frac{3}{4}$. Thus $x = n + h = 4 + \frac{3}{4} = \frac{19}{4} = 4.75$.

7. Find the number of terminating zeros the number $(100!)(50^{50})$ has after being multiplied out. (For example, the number 503,000,000 has 6 terminating zeros.)

   Answer: 124

   Solution: The number of terminating zeros is the same as the number of factors of 10, i.e., the number of times both 2 and 5 occur in the factorization. We have

$$50^{50} = (2 \cdot 5^2)^{50} = (2^{50})(5^{100})$$

and

$$100! = (100)(99)(98) \cdots (2)(1).$$

Let's count factors of 5 first. There are 100 factors of 5 in 50^{50}. To count the factors of 5 in 100!, note that the set of integers between 1 and 100 inclusive that are divisible by 5 is

$$S_1 = \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, 100\}$$

which has $|S_1| = [100/5] = 20$ elements, where $[x]$ denotes the greatest integer less than or equal to x. The set of integers between 1 and 100 inclusive that are divisible by 5^2 is

$$S_2 = \{25, 50, 75, 100\},$$

which has $|S_2| = [20/5] = 4$ elements.

There are no integers between 1 and 100 inclusive that are divisible by 5^n for $n \geq 3$. Thus the number of factors of 5 in 100! is $|S_1| + |S_2| = 24$. Adding this to the 100 factors of 5 from 50^{50}, we get a total of 124 factors of 5 in $(100!)(50^{50})$. If we can also find at least 124 factors of 2, then $(100!)(50^{50})$ will have 124 terminating zeros.

Let's count the factors of 2. There are clearly 50 factors of 2 in 50^{50}. Let T_n be the set of integers between 1 and 100 inclusive that are divisible by 2^n, and let $|T_n|$ be the number of elements in T_n. Then

$$\begin{aligned}
T_1 &= \{2, 4, 6, 8, ..., 100\}, \quad |T_1| = [100/2] = 50, \\
T_2 &= \{4, 8, 12, ..., 100\}, \quad |T_2| = [50/2] = 25, \\
T_3 &= \{8, 16, 24, ..., 96\}, \quad |T_3| = [25/2] = 12, \\
T_4 &= \{16, 32, 48, ..., 96\}, \quad |T_4| = [12/2] = 6, \\
T_5 &= \{32, 64, 96\}, \quad |T_5| = [6/2] = 3, \\
T_6 &= \{64\}, \quad |T_6| = [3/2] = 1.
\end{aligned}$$

So the number of factors of 2 in 100! is

$$\sum_{n=1}^{6} |T_n| = 50 + 25 + 12 + 6 + 3 + 1 = 97.$$

Thus the total number of factors of 2 in $(100!)(50^{50})$ is 50+97=147, which is more than enough to match the 124 factors of 5. Hence there are 124 factors of 10 in $(100!)(50^{50})$, which implies 124 terminating zeros.

8. Two trains head towards each other on the same straight track. The 1st train has a constant speed of 45 km/h and the 2nd train has a constant speed of 30 km/h. When the trains are 50 km apart, a bird flies from the front of the 1st train towards the 2nd, at a constant speed of 60 km/h. When the bird reaches the 2nd train, it immediately switches direction and flies back towards the 1st train. The bird repeatedly flies back and forth between the two trains, always flying at a constant speed of 60 km/h.

   (a) How many trips can the bird make from one train to the other before the two trains collide?

   (b) What is the total distance the bird travels?

   Answer: (a) infinity (b) 40 km

   Solution: Let's work part (b) first.The distance between the trains decreases at a constant rate of

$$45 \text{ km/h} + 30 \text{ km/h} = 75 \text{ km/h}.$$

   Starting out 50 km apart, the trains will collide after

$$\frac{50 \text{ km}}{75 \text{ km/h}} = \frac{2}{3} \text{ h}.$$

   In this time the bird will travel a total distance of

$$(60 \text{ km/h})(\frac{2}{3} \text{ h}) = 40 \text{ km}.$$

   For part (a), let's try computing the time for each trip and then add them up to see how many trips the bird can take in the $\frac{2}{3}$ h before the trains collide. To generalize our analysis, let u, v, and w denote the speeds of train 1, train 2, and the bird, respectively, and let s_0 denote the initial distance between the trains. In our problem,

$$u = 45 \text{ km/h}, \quad v = 30 \text{ km/h}, \quad w = 60 \text{ km/h}, \quad s_0 = 50 \text{ km}.$$

   Let t_n denote the duration of the bird's nth trip (for odd n the bird flies from train 1 to train 2, for even n the bird flies from train 2 to train 1).

   On the 1st trip (bird flies from train 1 to train 2), the distance from the bird to train 2 is initially s_0 and decreases at a constant rate of $w + v$. Thus

$$t_1 = \frac{s_0}{w + v}.$$

   On the 2nd trip (bird flies from train 2 to train 1), the distance from the bird to train 1 is initially $s_1 = (w - u)t_1$ (the difference between the distances that the bird and train 1 traveled during trip 1), and this distance decreases at a constant rate of $w + u$. Thus

$$t_2 = \frac{s_1}{w + u} = \frac{w - u}{w + u} t_1.$$

   On the 3rd trip (bird flies from train 1 to train 2), the distance from the bird to train 2 is initially $s_2 = (w - v)t_2$ (the difference between the distances that the bird and train 2 traveled during trip 2), and this distance decreases at a constant rate of $w + v$. Thus

$$t_3 = \frac{s_2}{w + v} = \frac{w - v}{w + v} t_2.$$

   Continuing with simalar arguments, we obtain the general result that for any integer $n \geq 2$,

$$t_n = \begin{cases} (\frac{w-v}{w+v})t_{n-1} & \text{for } n \text{ odd} \\ (\frac{w-u}{w+u})t_{n-1} & \text{for } n \text{ even.} \end{cases}$$

Combining the previous results, we obtain the recursion relation for any integer $n \geq 3$:

$$t_n = rt_{n-2}, \quad t_1 = \frac{s_0}{w+v}, \quad t_2 = \frac{w-u}{(w+u)(w+v)}s_0,$$

where

$$r = \frac{(w-u)(w-v)}{(w+u)(w+v)}.$$

Iterating the recursion relation (treating odd and even terms separately) yields, for any integer $j \geq 2$,

$$t_{2j-1} = t_1 r^{j-1}, \quad t_{2j} = t_2 r^{j-1}.$$

We conjecture that the bird can make an infinite number of trips before the trains collide. To verify this, we compute the sum:

$$\begin{aligned}
\sum_{n=1}^{\infty} t_n &= \sum_{j=1}^{\infty} t_{2j-1} + \sum_{j=1}^{\infty} t_{2j} \\
&= \sum_{j=1}^{\infty} t_1 r^{j-1} + \sum_{j=1}^{\infty} t_2 r^{j-1} \\
&= (t_1 + t_2) \sum_{j=1}^{\infty} r^{j-1} \\
&\overset{*}{=} (t_1 + t_2) \left(\frac{1}{1-r} \right) \\
&= \left(\frac{s_0}{w+v} + \frac{w-u}{(w+u)(w+v)} s_0 \right) \left(\frac{1}{1 - \frac{(w-u)(w-v)}{(w+u)(w+v)}} \right) \\
&= \frac{s_0}{u+v} \\
&= \frac{50 \text{ km}}{45 \text{ km/h} + 30 \text{ km/h}} \\
&= \frac{2}{3} \text{ h},
\end{aligned}$$

which is equal to the total time before the two trains collide as found in the solution to part (b). Thus the bird can make an infinite number of trips before the trains collide.

Note $(*)$: We used the geometric series result: $1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$ when $|r| < 1$.

Similar to the computation above, we can sum the distances traveled by each train on each of the bird's trips to verify that

$$\sum_{n=1}^{\infty} (u+v) t_n = s_0.$$

9. Suppose there are 18 socks, 5 of which are black, 6 of which are brown, and 7 of which are gray. You pick two socks out at a time (sampling without replacement), and each set of two forms a pair. So you just form 9 pairs of socks at random without worrying about matching. What is the expected value of the number of matching pairs? In other words, if you repeated this experiment a very large number of times, what would be the average number of matching pairs?

Answer: $\frac{46}{17}$

Solution: The expected value of a random variable X that takes discrete values x_1, x_2, x_3, ..., with probabilities $P(X_1 = x_1)$, $P(X_2 = x_2)$, $P(X_3 = x_3)$, ..., respectively, is

$$E(X) = \sum_i x_i P(X_i = x_i).$$

For $i = 1, 2, ..., 9$, define the random variable X_i by

$$X_i = \begin{cases} 1 & \text{if the } i\text{th pair matches} \\ 0 & \text{if the } i\text{th pair doesn't match.} \end{cases}$$

Then the total number of matching pairs is given by the random variable X defined by

$$X = \sum_{i=1}^{9} X_i.$$

First let's compute the expected value of X_i, for $i = 1, 2, ..., 9$:

$$\begin{aligned} E(X_i) &= (1)P(X_i = 1) + (0)P(X_i = 0) \\ &= P(X_i = 1). \end{aligned}$$

So we need to find $P(X_i = 1)$, the probability that the ith pair is a match. The total number of ways to choose 2 socks from 18 is

$$\binom{18}{2} = \frac{18!}{(2!)(16!)} = \frac{(18)(17)}{2} = (9)(17) = 153.$$

For the ith pair to match, it must have either 2 black, 2 brown, or 2 gray socks.

The number of ways of choosing 2 black socks from the 5 available is

$$\binom{5}{2} = \frac{5!}{(2!)(3!)} = \frac{(5)(4)}{2} = 10.$$

The number of ways of choosing 2 brown socks from the 6 available is

$$\binom{6}{2} = \frac{6!}{(2!)(4!)} = \frac{(6)(5)}{2} = 15.$$

The number of ways of choosing 2 gray socks from the 7 available is

$$\binom{7}{2} = \frac{7!}{(2!)(5!)} = \frac{(7)(6)}{2} = 21.$$

Thus there are $10+15+21 = 46$ possible ways the ith pair can match, out of a total of 153 possibilities, so the probability that the ith pair is a match is

$$P(X_i = 1) = \frac{46}{153}.$$

Therefore, for $i = 1, 2, ..., 9$,

$$E(X_i) = P(X_i = 1) = \frac{46}{153}.$$

The expected value of the total number of matching pairs is then

$$E(X) = E\left(\sum_{i=1}^{9} X_i\right) = \sum_{i=1}^{9} E(X_i) = \sum_{i=1}^{9} \frac{46}{153} = 9\left(\frac{46}{153}\right) = \frac{46}{17}.$$

An alternate approach for computing $P(X_i = 1)$ uses conditional probability. Let $P(A|B)$ denote the probability of event A given that event B has occurred. This is defined as $P(A|B) = \frac{P(A \cap B)}{P(B)}$.
Let the random variables Y_1 and Y_2 represent the colors of each sock in the ith pair, and let C denote the set of possible colors:

$$C = \{\text{black, brown, gray}\}.$$

Then the probability that the ith pair is a match can be written as

$$\begin{aligned} P(X_i = 1) &= \sum_{c \in C} P((Y_1 = c) \cap (Y_2 = c)) \\ &= \sum_{c \in C} P(Y_2 = c | Y_1 = c) P(Y_1 = c) \\ &= \left(\frac{4}{17}\right)\left(\frac{5}{18}\right) + \left(\frac{5}{17}\right)\left(\frac{6}{18}\right) + \left(\frac{6}{17}\right)\left(\frac{7}{18}\right). \end{aligned}$$

As before, the expected value of the total number of matching pairs is then

$$\begin{aligned} P(X) &= \sum_{i=1}^{9} E(X_i) \\ &= \sum_{i=1}^{9} P(X_i = 1) \\ &= 9P(X_i = 1) \\ &= \frac{(4)(5) + (5)(6) + (6)(7)}{(17)(2)} \\ &= \frac{46}{17}. \end{aligned}$$

10. Consider a $1 \times n$ array of squares covered by tiles that are each 1×1. Each tile is either blue, red, or yellow. Yellow tiles always occur at least two in a row. There are no restrictions on the number of blue or red tiles that are consecutive. Here are some examples of sequences that satisfy the constraints:

$$brbbyyr \quad yybyyryyrr \quad byyrryyyrrb \quad bbrrbbrbr$$

And some that don't

$$yybbrry \quad brrbry$$

Find the number of sequences that satisfy the constraints for $n = 7$ squares.

Answer: 527

Solution: Let $f(n)$ denote the number of sequences that satisfy the given conditions for n tiles. We consider some specific cases for n.

Case $n = 1$: There are 2^1 sequences without yellow (b, r), and 0 sequences that have yellow. Thus

$$f(1) = 2 + 0 = 2.$$

Case $n = 2$: There are 2^2 sequences without yellow (bb, rr, br, rb) and 1 sequence with yellow (yy). Thus

$$f(2) = 4 + 1 = 5.$$

Case $n = 3$: There are 2^3 sequences without yellow (bbb, bbr, brb, rbb, rrb, rbr, brr, rrr), 4 sequences with 2 yellow (yyb, yyr, byy, ryy) and 1 sequence with 3 yellow (yyy). Thus

$$f(3) = 8 + 4 + 1 = 13.$$

Case $n = 4$: There are 2^4 sequences without yellow, 12 sequences with 2 yellow ($yyxx$, $xyyx$, $xxyy$), 4 sequences with 3 yellow ($yyyx$, $xyyy$) and 1 sequence with 4 yellow ($yyyy$), where each x could be r or b. Thus

$$f(4) = 16 + 12 + 4 + 1 = 33.$$

Now let's look for a recursion relation. Suppose $f(1)$, $f(2)$, ..., $f(n-1)$ have been determined. Consider a sequence with n tiles. If the nth tile is r or b, then the previous $n-1$ tiles must be a sequence that satisfies the given conditions. Thus there are $f(n-1)$ valid sequences of n tiles that end in r, and another $f(n-1)$ valid sequences of n tiles that end in b.

To aid in visualization, we let r denote red, b denote blue, y denote yellow, x could be red or blue, and z could be red, blue, or yellow (assuming the required conditions are satisfied). We use subscripts to indicate order of the tiles in a sequence. With this notation, an n tile sequence that ends in red or blue has the form

$$z_1 z_2 z_3 \ldots z_{n-3} z_{n-2} z_{n-1} x_n,$$

and there are $2f(n-1)$ such sequences that satisfy the required conditions.

If an n tile sequence ends in y, then both the nth and $(n-1)$th tiles must each be y. Such sequences have the form

$$z_1 z_2 z_3 \ldots z_{n-3} z_{n-2} y_{n-1} y_n.$$

Looking at only the first $n-2$ tiles of such a sequence, either z_{n-2} is an isolated y (meaning z_{n-2} is y and z_{n-3} is not y), or z_{n-2} is not an isolated y (meaning either z_{n-2} is not y or z_{n-2} and z_{n-3} are both y). In the latter case (z_{n-2} not an isolated y), the first $n-2$ tiles form a valid $(n-2)$ tile sequence, hence there are $f(n-2)$ of these sequences. In the former case (z_{n-2} is an isolated y), the sequence has the form

$$z_1 z_2 z_3 \ldots z_{n-4} x_{n-3} y_{n-2} y_{n-1} y_n.$$

Since the first $n-4$ tiles of such a sequence must satisfy the required conditions, and x_{n-3} could be r or b, there are $2f(n-4)$ sequences of this form.

To summarize, from the set of valid n tile sequences, $2f(n-1)$ of them end in r or b, and $f(n-2)+2f(n-4)$ of them end in y. Thus we obtain the recursion relation for $n \geq 5$:

$$f(n) = 2f(n-1) + f(n-2) + 2f(n-4), \quad f(1) = 2, \quad f(2) = 5, \quad f(3) = 13, \quad f(4) = 33.$$

We can then easily compute:

$$\begin{aligned} f(5) &= 2f(4) + f(3) + 2f(1) = 2(33) + 13 + 2(2) = 83 \\ f(6) &= 2f(5) + f(4) + 2f(2) = 2(83) + 33 + 2(5) = 209 \\ f(7) &= 2f(6) + f(5) + 2f(3) = 2(209) + 83 + 2(13) = 527 \end{aligned}$$

Thus the number of sequences with $n = 7$ tiles that satisfy the required conditions is 527.

An alternate method of solution is to use brute force counting for 7 tiles, based on how many yellow tiles are in the sequence. In some of the cases below we make use of the fact that given positive integers k and m, the number of solutions in $\{0, 1, 2, 3, \ldots\}$ of the equation $x_1 + x_2 + \cdots + x_k = m$ is $C(m+k-1, k-1) = \frac{(m+k-1)!}{(k-1)!(m!)}$.

- 0 yellow: $2^7 = 128$ sequences (each of the 7 tiles is either red or blue).
- 1 yellow: 0 sequences (can't have just 1 yellow).
- 2 yellow: $6(2^5) = 192$ sequences (6 choices for where yy goes, 2^5 choices of r or b for 5 remaining tiles).
- 3 yellow: $5(2^4) = 80$ sequences (5 choices for where yyy goes, 2^4 choices of r or b for 4 remaining tiles).
- 4 yellow: $10(2^3) = 80$ sequences. They have the form $X_1yyX_2yyX_3$, where X_i denotes a block of x_i tiles that are r or b, where each $x_i \in \{0, 1, 2, 3\}$ and $x_1 + x_2 + x_3 = 3$. There are $C(5, 2) = \frac{5!}{(2!)(3!)} = 10$ solutions of this equation and 2^3 choices for the 3 non-yellow tiles.
- 5 yellow: $9(2^2) = 36$ sequences. One type has the form $X_1yyX_2yyyX_3$, where X_i denotes a block of x_i tiles that are r or b, where each $x_i \in \{0, 1, 2\}$ and $x_1 + x_2 + x_3 = 2$. There are $C(4, 2) = \frac{4!}{(2!)(2!)} = 6$ solutions of this equation. The other type has the form $X_1yyyX_2yyX_3$, which clearly has the same number of sequences. Including both types double counts the sequences with $x_2 = 0$, i.e. sequences of the form $X_1yyyyyX_3$, where $x_1 + x_3 = 2$, which has $C(3, 1) = \frac{3!}{(1!)(2!)} = 3$ solutions. Thus we have $2(6) - 3 = 9$ choices for the placement of yellow tiles and 2^2 choices for the remaining 2 non-yellow tiles.
- 6 yellow: $(5)(2^1) = 10$ sequences. One type has the form $X_1yyX_2yyX_3yyX_4$ where X_i denotes a block of x_i tiles that are r or b, where each $x_i \in \{0, 1\}$ and $x_1 + x_2 + x_3 + x_4 = 1$. There are $C(4, 3) = \frac{4!}{(3!)(1!)} = 4$ solutions of this equation. The other type has the form $X_1yyyX_2yyyX_3$ where $x_i \in \{0, 1\}$ and $x_1 + x_2 + x_3 = 1$. There are $C(3, 2) = \frac{3!}{(2!)(1!)} = 3$ solutions of this equation. Including both types double counts the sequences of the form $X_1yyyyyX_3$, where $x_1 + x_3 = 1$, which has $C(2, 1) = \frac{2!}{(1!)(1!)} = 2$ solutions. Thus we have $4 + 3 - 2 = 5$ choices for the placement of yellow tiles and 2^1 choices for the 1 remaining non-yellow tile.
- 7 yellow: $(1)(2^0) = 1$ sequence (namely $yyyyyyy$).

The total number of 7 tile sequences satisfying the required conditions is thus

$$128 + 0 + 192 + 80 + 80 + 36 + 10 + 1 = 527.$$