UNM–PNM STATEWIDE MATHEMATICS CONTEST XXXVI

SECOND ROUND SOLUTIONS

We begin with a beautiful solution, provided by **L.S. Hahn**, of part of Problem 8 from the Fall 2003 UNM–PNM Math contest which asked the following:

8. Farmer Brown has eight logs, each of length 10 feet. What is the *maximum* area which he can enclose with the logs? For example, he could make a rectangle of height 10 feet and width 30 feet or a square with each side having length 20 feet. In the first case, he has enclosed 300 square feet and in the second case 400 square feet. The second choice is of course better than the first but what is the *largest* area which Farmer Brown can enclose?

It was stated in the solution that the maximum enclosed area will come from a regular octagon. Suppose that you have an octagon $ABCDEFGH$ of maximal area. First, it is sufficient to consider the case where the octagon is *convex*, that is any line segment joining two points in the octagon lies entirely within the octagon. To see this, suppose the octagon fails to be convex, say, at the vertex B. Then the angle ABC can be flipped outwards and this will *increase* the total area, violating the hypothesis that $ABCDEFGH$ had maximal area.

Next, suppose we draw the line segment AE, splitting the octagon into two pentagons. Each of these pentagons must have maximal area, under the constraint that four of its sides have fixed length (namely 10 feet), or else we could construct an octagon of *larger* area.

Consider now the angle ACE. We claim that this is a right angle. Indeed, viewing the triangles ABC and CDE as fixed and allowing the angle ACE to vary, we know that the triangle ACE must have maximal area because the pentagon ABCDE has maximal area and this pentagon is the sum of the two fixed triangles and the triangle ACE. Thus we have a triangle ACE with the lengths of AC and CE fixed and we would like to maximize its area. This happens precisely when ACE is a right angle (this is a good exercise).

It follows then that the point C is on the circle having AE as its diameter. A similar argument applies to show that the angles ABE, ADE, EFA, EGA, and EHA are right angles. Thus all eight vertices of the octagon lie on a circle with diameter AE. Since all the sides must have equal length (10 feet), this shows that the octagon is regular.

1. We'll begin with the required problem about the current year 2004.

a. Give the prime factorization of 2004.

b. How many positive integers divide 2004 evenly?

We find, for part **a**, that $2004 = 2^2 \cdot 3 \cdot 167$.

As for part **b** a divisor of 2004 will look like

$$2^a 3^b 167^c$$

where $0 \le a \le 2$, $0 \le b \le 1$, $0 \le c \le 1$. Thus there are 3 choices for a and two choices for b and c making a total of 12 divisors of 2004. Of course, one can also solve this problem by finding the divisors explicitly.

2. Here is another problem concerning prime numbers.

a. Let $L(X) = 12X + 115$. Find the *smallest* integer $n \ge 1$ so that $L(n)$ is *not* a prime number.

b. Does there exist an integer $a \ge 1$ so that $an + 1$ is a prime number for all $n \ge 1$? Justify your answer.

c. Suppose $P(X) = a_d X^d + a_{d-1} X^{d-1} + \ldots + a_1 X + a_0$ is a polynomial where the coefficients $a_0, \ldots, a_d$ are integers, $a_d \ne 0$, and $d \ge 1$. Can $P(n)$ be a prime number for *all* integers $n \ge 1$ when the degree d is arbitrary? Justify your answer.

For part **a** we find that $L(5) = 175$ is not a prime number while $L(1), L(2), L(3)$, and $L(4)$ are each prime.

The answer to **b** is NO: consider $n = a + 2$. We find $an + 1 = a^2 + 2a + 1 = (a + 1)^2$ which is never a prime number since $a \ge 1$.

The answer to **c** is also NO but this is a little trickier to see. If a_0 is not 0 or ± 1 then we see that $P(ka_0)$ is not prime for k large: indeed, a_0 divides $P(ka_0)$ and must be a proper divisor if k is large since $P(X)$ approaches plus or minus infinity as X grows. If $a_0 = 0$ then X divides $P(X)$ and so for X large $P(X)$ will never be prime. Suppose then that $a_0 = \pm 1$. Consider the polynomial $Q(X) = P(X + n)$. Since $Q(0) = P(n)$ we can assume that $Q(0)$ is not $0, 1$, or -1. Arguing as before $Q(mQ(0))$ must be composite when m is sufficiently large but $Q(mQ(0)) = P(mQ(0) + n)$.

Another way to solve **b**, suggested by **Jacob Hobbs** and **Ila Varma** of La Cueva High School and also by **Rima Turner** of Los Alamos High School, is the following. Suppose that $an + 1$ is a prime number whenever n is positive. Then we claim that a is even. Indeed, if a is odd then $3a + 1$ is even and larger than 2 and consequently is not prime. Next we claim that 3 must divide a. If not, then there are two possibilities: either a leaves a remainder of 1 when divided by 3 (in which case $5a + 1$ is properly divisible by 3 and hence not prime) or else a leaves a remainder of 2 when divided by 3 (in which case $4a + 1$ is properly divisible by 3).

In general, we claim that if p is *any* prime number then p must divide a. Indeed suppose p does not divide a. Then we can always find $b > p$ so that ab leaves a remainder of $p - 1$ when divided by p. But then $ab + 1$ is properly divisible by p and hence not prime. As to the existence of $b > p$ so that $ab+1$ is divisible by p, consider the numbers $a(p+1), a(p+2), \ldots, a(2p-1)$. We see that none of these numbers, or their differences, is divisible by p. Thus they must leave *different* remainders when divided by p. But there are only $p-1$ possible remainders and $p-1$ numbers on our list and this establishes the existence of b as claimed.

Of course a cannot be divisible by every prime number and thus we have a contradiction and it cannot be the case that $an+1$ is always prime.

Finally, **Bob Cordwell** of Manzano High School was the only person to solve part (c) correctly (his solution was essentially identical to the one presented above).

3. The sequence of Fibonacci numbers is:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

The first two elements of the sequence are 1 and then each successive member is obtained by adding the two previous elements: $F_n = F_{n-1} + F_{n-2}$ where F_n denotes the n^{th} number in the sequence.

a. Find F_{17}, the 17^{th} Fibonacci number.

b. Show that

$$\frac{F_2}{F_1} < \frac{F_4}{F_3} < \frac{F_6}{F_5} < \ldots,$$

i.e. show that for any $n > 0$ we always have $F_{2n}/F_{2n-1} < F_{2n+2}/F_{2n+1}$.

c. What value must

$$\frac{F_{n+1}}{F_n}$$

approach as n grows, *assuming that it does approach some value*?

Part **a** is a purely computational problem and the answer is 1597.

For part **b** we will use the following fact: suppose a, b, c, d are positive integers and $a/b < c/d$. Then

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

One can prove these inequalities by cross–multiplying: for example $\frac{a}{b} < \frac{a+c}{b+d}$ if and only if $ab + ad < ba + bc$ which is true by hypothesis. We use this then to consider the successive fractions F_{n+1}/F_n. We start with $F_2/F_1 = 1/1$ and $F_3/F_2 = 2/1$. In particular since

$$\frac{F_2}{F_1} < \frac{F_3}{F_2}$$

we conclude from above that

$$\frac{F_2}{F_1} < \frac{F_2+F_3}{F_1+F_2} = \frac{F_4}{F_3} < \frac{F_3}{F_2}.$$

To make the next step, we add the last two fractions in the above equation to find

$$\frac{F_4}{F_3} < \frac{F_4 + F_3}{F_3 + F_2} = \frac{F_5}{F_4}$$

and adding one more time gives what we want

$$\frac{F_4}{F_3} < \frac{F_6}{F_5} < \frac{F_5}{F_4}.$$

It is then clear that this procedure can continue ad infinitum.

For part **c** note that we have the relation

$$\frac{F_{n+2}}{F_{n+1}} = 1 + \frac{F_n}{F_{n+1}}.$$

Letting α be the number which $\frac{F_{n+1}}{F_n}$ approaches as n grows, we see that

$$\alpha = 1 + \frac{1}{\alpha}$$

or $\alpha^2 - \alpha - 1 = 0$. Thus

$$\alpha = \frac{1 \pm \sqrt{5}}{2}$$

and since $\alpha \geq 1$ we must have $\alpha = \frac{1+\sqrt{5}}{2}$.

One very interesting solution to parts (b) and (c) of this problem was given by **Zach Labry** from the Albuquerque Academy. He uses the Binet formula which gives a closed expression for the N^{th} Fibonacci number, namely

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^N - \left(\frac{1-\sqrt{5}}{2}\right)^N}{\sqrt{5}}$$

Letting $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ we see, for part (c), that

$$\frac{F_{n+1}}{F_n} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n}.$$

Since $|\beta^n|$ gets closer and closer to 0 as n grows, we see that

$$\frac{F_{n+1}}{F_n}$$

gets closer and closer to

$$\frac{\alpha^{n+1}}{\alpha^n} = \alpha$$

and this establishes part (c).

The Binet formula can also be used nicely to prove part (b). In particular we need to show
$$\frac{\alpha^{2n}-\beta^{2n}}{\alpha^{2n-1}-\beta^{2n-1}} < \frac{\alpha^{2n+2}-\beta^{2n+2}}{\alpha^{2n+1}-\beta^{2n+1}}.$$
Cross–multiplying and simplifying, this is the same as showing that

$$\alpha^{2n+2}\beta^{2n-1}+\alpha^{2n-1}\beta^{2n+2} < \alpha^{2n+1}\beta^{2n}+\alpha^{2n}\beta^{2n+1}.$$

We now divide through this equation by $\alpha^{2n-1}\beta^{2n-1}$. Since this is a negative number, this will reverse the inequality sign giving

$$\alpha^2\beta+\alpha\beta^2 < \alpha^3+\beta^3.$$

The important thing to note here is that the variable n has disappeared and we now have a simple inequality of real numbers which we leave as an exercise.

Finally, **Jacob Hobbs** of La Cueva High School and **Dimitar Bounov** of United World College provided the following beautiful inductive proof of part (b). We wish to show that
$$\frac{F_{2n}}{F_{2n-1}} < \frac{F_{2n+2}}{F_{2n+1}}$$
when $n \geq 1$. For $n = 1$, this is a simple calculation so assume that $n \geq 2$. Then we have
$$\frac{F_{2n+2}}{F_{2n+1}} = 1 + \frac{F_{2n}}{F_{2n+1}}$$
and
$$\frac{F_{2n}}{F_{2n-1}} = 1 + \frac{F_{2n-2}}{F_{2n-1}}$$
So $\frac{F_{2n}}{F_{2n-1}} < \frac{F_{2n+2}}{F_{2n+1}}$ if and only if
$$\frac{F_{2n-2}}{F_{2n-1}} < \frac{F_{2n}}{F_{2n+1}}$$
Taking the reciprocal this is equivalent to

$$\frac{F_{2n-1}}{F_{2n-2}} > \frac{F_{2n+1}}{F_{2n}}$$

and then iterating the above argument shows that this last inequality holds if and only if
$$\frac{F_{2n-2}}{F_{2n-3}} < \frac{F_{2n}}{F_{2n-1}}$$
which is true by the inductive hypothesis.

4. David tosses a quarter in the air and watches it land on a checker board. Suppose that the length and width of each of the 64 squares in the checker board is exactly *twice* the diameter of the quarter and also suppose that the quarter lands *entirely* within the checker board, with equal probability at any point.

a. What is the probability that the quarter lands entirely within one of the 64 squares?

b. What is the probability that the quarter touches exactly three of the 64 squares?

Suppose for the moment that the diameter of the quarter is one inch and the squares in the checker board are two inches by two inches (the units of measure make no difference in the answer to this question). For a given square, the area is 4 square inches and in order for the quarter to land entirely within the square this would mean that the *center* of the quarter has to land within a one inch by one inch square, centered in the two by two square. So the total area of good throws is 64 square inches. The total area of *allowed* throws, on the other hand, is 225 square inches because the quarter must land entirely within the checkerboard and this eliminates a one half inch band all the way around the board. So the answer here is 64/225.

To find the probability that the quarter touches exactly two squares, note that the two squares must be adjacent. Counting up the number of adjacent squares, we find 7 in each of the 8 rows making 56 and then 7 more in each of the 8 columns for 56 more. Thus there are 112 adjacent squares in which the quarter could land. For each pair of adjacent squares there is an inner square which is one inch by one inch in which the center of the quarter can land if it is contained entirely within these two squares and touches both. Thus the probability of landing on exactly two squares is $\frac{112}{225}$.

To find the probability that the quarter touches 4 squares, note that there are 49 vertices on the board where 4 squares meet. In order for the quarter to touch all 4 it must land within a circle of radius 1/2 centered at the vertex where the squares meet. Thus this is an area of $\frac{49\pi}{4}$ and the probability of touching 4 squares is $\frac{49\pi}{900}$.

The only remaining possibility is that the quarter touches 3 squares. Since it must touch exactly one, two, three, or four squares and since these possibilities are mutually exclusive, the probability that it touches exactly three squares is

$$1 - \frac{64}{225} - \frac{112}{225} - \frac{49\pi}{900} = \frac{196 - 49\pi}{900}.$$

5. Way back in 1901, Jacqueline's great grandfather deposited a brand new 1901 quarter from the San Francisco mint in the bank. This 1901 quarter is, however, a VERY rare coin, worth $40,000 today in 2004.

a. At 20% annual interest, will the quarter be worth more or less than one dollar after 8 years?

b. Assuming that the quarter earned 10% annual interest for the entire 103 year period, would it be worth more or less than $40,000?

c. Assuming that the quarter earned 13% annual interest for the entire 103 year period, would it be worth more or less than \$40,000?

Note that your answers in **b** and **c** must be justified– please explain clearly your method of calculating and how you plan to make the necessary computations without a calculator.

If an investment of P is made at an interest rate of $x\%$, compounded annually, then after t years the investment is worth

$$P\left(1+\frac{x}{100}\right)^t.$$

Thus for part **a** we need to compute $0.25(1.2)^8$. We find

$$\begin{aligned}(1.2)^2 &= 1.44,\\ (1.2)^4 &= (1.44)^2 > 2,\\ (1.2)^8 &= (1.2)^4(1.2)^4 > 4.\end{aligned}$$

So after eight years the quarter is worth *more* than one dollar.

For part **b** we try to find how long it takes for the quarter to double in value and then extrapolate from this.

$$\begin{aligned}(1.1)^2 &= 1.21,\\ (1.1)^4 &= (1.21)^2 < 1.5,\\ (1.1)^6 &< (1.21)(1.5) < 2.\end{aligned}$$

Thus each after each 6 years the quarter has gone up in value by a multiple of *less than two.* We have $102 = 17 \cdot 6$ so after 102 years the quarter is worth less than

$$\$(0.25 \cdot 2^{17}) = \$2^{15} = \$(1024)(32) < \$33,000.$$

Finally, after 103 years the quarter will worth less than

$$\$33,000 \cdot 1.1 = \$36,300.$$

Note that these estimates are far from the best possible but are sufficient to answer the question.

For part **c** we need to find a bound from below rather than from above as the quarter will end up being worth more than $\$40,000$.

$$\begin{aligned}(1.13)^2 &= 1.2769 > 1.276,\\ (1.13)^4 &= (1.13)^2(1.13)^2 > (1.276)^2 = 1.628176 > 1.628,\\ (1.13)^8 &= (1.13)^4(1.13)^4 > (1.628)(1.628) = 2.650384 > 2.65,\\ (1.13)^9 &= (1.13)^8(1.13) > 2.65(1.13) = 2.9945.\end{aligned}$$

Here we see that we have almost exactly 3 as a lower bound but not quite so we need to go back and improve the lower bounds a bit. So we find

$$\begin{aligned}
(1.13)^2 &= 1.2769,\\
(1.13)^4 &= (1.13)^2(1.13)^2 > (1.2769)^2 = 1.63047361 > 1.63,\\
(1.13)^8 &= (1.13)^4(1.13)^4 > (1.63)(1.63) = 2.6569 > 2.656,\\
(1.13)^9 &= (1.13)^8(1.13) > 2.656(1.13) = 3.00128.
\end{aligned}$$

Now we have

$$(1.13)^{99} > ((1.13)^9)^{11} > 3^{11}.$$

As for 3^{11} note that $3^4 = 81 > 80$ so

$$3^{11} > (80)(80)(27) > (80)(80)(25) = 80(2000) = 160,000.$$

Thus after 99 years the quarter is worth more than \$40,000 and thus after 103 years it is worth well more than \$40,000.

Note that the key here is to raise 1.3 to a power until you find something which is *close* to a whole number. In this case the power is not too large and so the computation is doable. You should check to see what the smallest power of 1.3 is which is bigger than 2 and see if taking powers of 2 will work instead of the powers of three which we did.

Only one student successfully solved this problem, **Clayton Shepard** of the Albuquerque Academy. His method was to successively square 1.1 (for part (b)) and 1.13 (for part (c)), rounding off whenever possible. Many others attempted similar computations but only Clayton was successful with the required estimates.

Carl Grover of La Cueva High School had the following very beautiful idea to do 5(b). Note that

$$\left(1 + \frac{1}{n}\right)^n$$

gets closer and closer to e as n grows. Moreover this quantity increases as n increases. In particular,

$$\left(1 + \frac{1}{10}\right)^{10} < e < 3.$$

Thus at 10% interest, after 100 years the quarter has increased in value by a factor of

$$\left(1 + \frac{1}{10}\right)^{100} < 3^{10}.$$

It is simple to get from here to a bound of less than \$40,000 for the value of the quarter after 103 years. This method does *not* work, however, for part (c) because in this case one needs a lower bound.

6. Your school teacher presents you with the following problem: she gives you a hat, containing 5 slips of paper, each with a different number on it. You know *nothing* about the numbers: they could be of any size, positive or negative. You are asked to draw numbers successively out of the hat and look at them. The problem is to stop at the moment you have selected the *largest* of the 5 numbers. Of course, since you do not know what any of the numbers are in advance, it is impossible to solve this problem with *certainty.*

a. Suppose you employ the following method: you look at and discard one number which we will call A. Next you continue to draw until you find a number *larger* than A and stop here. What is the probability that you have stopped at the largest of the five numbers? Note that the definition of probability here is the total number of cases in which you are successful divided by the total number of all possible cases.

b. Assuming now that there are 100 slips of papers in the hat, each with a different number on it, give a method which allows you to stop at the moment you have selected the largest number more than $1/4$ of the time. Of course you need to prove that the method will be successful more than $1/4$ of the time.

Note that the definition of the success rate of your method is the total number of possibilities in which it stops at the largest number divided by the total number of *all* possibilities successful or not.

For part **a**, we break it down into cases. Suppose we label the numbers

$$A < B < C < D < E.$$

So let's suppose the first number drawn is A. Here you will only win if E comes second and there are 6 such possibilities. Next let's suppose the first number drawn is B. Then you win if E comes next (6 cases) or if A comes next and then E (2 cases). The most difficult case to analyze is where the first number drawn is C. Here you can win if E is drawn second (6 cases) or E is drawn third while the second draw is A or B (4 cases) or if E is drawn fourth while the second and third choices are A and B (2 cases). Next, suppose D is drawn first. Then you win all 24 possible cases here. Finally, if E is drawn first then you lose. The total number of winning cases is

$$6 + 6 + 2 + 6 + 4 + 2 + 24 = 50.$$

So this strategy is successful $50/120 = 5/12$ of the time.

For part **b** a similar method works. You look at and discard fifty numbers and then stop as soon as you draw a number which is bigger than any of the first fifty (again, if you never find a larger number, then the method fails). As in the first part, call the second largest number B and the largest number A. Then B will occur within the first 50 numbers drawn exactly $\frac{1}{2}$ of the time: indeed each instance where B occurs within the first 50 has a mirror image, where the order is reversed, with B occurring in the second 50 selections. Assuming that B occurs within the first 50 draws, there is greater than 1/2 probability that A occurs in the second 50 draws (to be precise, there is a 50/99 probability). Thus this method successfully stops at A more than 1/4 of the time.

Bob Cordwell of Manzano High School, **Rima Turner** of Los Alamos High School, **Zach Labry** of the Albuquerque Academy, **Carl Grover** of La Cueva High School, **Brian Geistwhite** of Farmington High School, **Alan Hshieh** of Las Cruces High School, **Jacob Hobbs** of La Cueva High School, and **Adolf James** of the Albuquerque Academy were able to shorten the argument for part (a) substantially as follows. If A is drawn first then there is a 1/4 probability of stopping with E. If B is drawn first there is a 1/3 probability of stopping with E. If C is drawn first there is a 1/2 probability of stopping with E and so on. Adding things up gives 5/12 as above but not as much counting is necessary.

7. An integer–valued point in the xy–plane is a point (a, b) where both a and b are integers. Let A_n denote the number of integer–valued points on or inside a circle of radius n centered at the origin. As n grows larger and larger what value will

$$\frac{A_n}{n^2}$$

approach? Justify your answer.

As you may be able to guess from the similar question on the Fall exam, the answer here is π. To see this, suppose n is some large number and let C_1 be a circle of radius $n - 2$ centered at the origin and C_2 a circle of radius $n + 2$. To each integer point (x, y) contained on or inside C_1 we associate the square, with side of length one, whose southwest corner is placed at (x, y). All of these squares are entirely contained within C, the circle of radius n and so their total area is at most the area of C. Thus if b_n is the number of integer points contained inside of C_1 we find that

$$b_n \leq \pi n^2.$$

By definition $a_n = b_{n+2}$ so

$$\frac{a_n}{(n+2)^2} = \frac{b_{n+2}}{(n+2)^2} \leq \pi.$$

It follows that

$$\frac{a_n}{n^2} \leq \frac{\pi(n+2)^2}{n^2}$$

and thus as n grows we find that

$$\frac{a_n}{n^2}$$

can approach a value of *at most* π.

For the opposite inequality, let c_n be the number of integer valued points in C_2. Then we see that $c_n \geq \pi n^2$ since the entire circle of radius n is covered by the squares placed, as before, at integer points in C_2: to see this suppose that $P \in C$ is not contained in *any* square whose southwest corner is at an integer valued point of C_2. Let S be the unique unit square, with integer valued coordinates, containing P and so that P is to the northeast of its soutwest corner. Call the southwest corner of this square (a, b). Then the point (a, b) has a distance of at most $\sqrt{2}$ from P and so is a point in C_2 and we have reached the desired contradiction. As above we have $a_{n+2} = c_n$ and so

$$\frac{a_{n+2}}{(n+2)^2} = \frac{c_n}{(n+2)^2} \geq \frac{\pi n^2}{(n+2)^2}.$$

Arguing as above shows that if $\frac{a_n}{n^2}$ approaches a value then this value must be *at least* π. Combining this with the first part of the argument above we find that $\frac{a_n}{n}$ must approach π as n grows.

The only student in the contest to provide a solution like the one above was **Jeff Dimiduk** of Eldorado High School. An interesting variant was provided by **Bob Cordwell** of Manzano High School. In particular, Bob bounds the number of squares which the circle can touch but not contain. In particular, the circle of radius n has circumference $2n\pi$. On the other hand, a segment of length < 1 of this circumference can intersect *at most* 4 squares. Thus the number of squares which the entire circumference can intersect is at most $8n\pi + 1$. We leave as an exercise how to conclude the argument from this point.