UNM-PNM STATEWIDE MATHEMATICS CONTEST XXXVII

FEBRUARY 5th, 2005 SECOND ROUND THREE HOURS

PROBLEM 1: Abran, Alisa, Ava and Alejandro are walking home in the middle of the night. It is very dark and they only have one lantern. They have to cross a wooden bridge. The bridge is in very poor condition and can support at most two of them at a time. Those crossing the bridge need the lantern so as not to fall through the cracks in the old wood, therefore the lantern needs to be transported back and forth until all of them have crossed the bridge.

Alisa is very fast and can cross the bridge in 1 minute. Abran is also quite fast and can cross it in 2 minutes. Ava is less fast but can still do it in 5 minutes. Finally Alejandro, who is very scared of heights and of the darkness, needs at least 10 minutes to cross the bridge.

We know the bridge will collapse after 18 minutes of walking over it. Will all of them be able to cross the bridge safely? If YES, describe how, and in how many minutes they can cross. If NOT, explain why and tell us what is the least amount of minutes in which all four friends can cross safely.

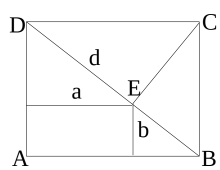
PROBLEM 2: Let x, y, z be real numbers. Suppose $(x+1)(y+1)(z+1) \neq 0$, and

$$\frac{x}{x+1} + \frac{y}{y+1} + \frac{z}{z+1} = 1.$$

Find all possible values of the quantity $(2xyz + xy + yz + zx)$ for all x, y, z with the above properties.

PROBLEM 3: Amy has the following rule to distribute candies on Halloween: the first child to come receives a 23rd of the candies plus one candy, the second one receives a 23rd of the remaining candies plus two candies, the third one receives a 23rd of the remaining candies plus three candies, etc. All the candies were given away and all the kids received the same amount of candies. How many children visited Amy? How many candies did each child get?

PROBLEM 4: Suppose E is the foot of the perpendicular from C to diagonal BD in rectangle $ABCD$. If the lengths of perpendiculars from E to AD and AB are a and b, respectively, express the length d of diagonal BD in terms of a and b.



PROBLEM 5: Remember that $\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$.

For example $\sum_{k=1}^{7} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$, in this case $n = 7$ and $a_k = k^2$.

(a) Evaluate $\displaystyle\sum_{k=1}^{5} \frac{1}{k(k+1)(k+2)}$.

(b) Find an integer m in terms of n such that $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{4} - \frac{1}{m}$.

PROBLEM 6: **(a)** Given 6 points on a circle, how many chords are there having two of these 6 points as endpoints? What is the maximum possible number of intersections these chords can make in the interior of the circle? What is the maximum possible number of regions these chords can divide the interior of the circle?

(b) Given 12 points on the circle, how many chords are there having two of these 12 points as endpoints? What is the maximum possible number of intersections these chords can make in the interior of the circle? What is the maximum possible number of regions these chords can divide the interior of the circle?

PROBLEM 7: Let ABC be an acute triangle. Recall that an acute triangle has all angles less than $90°$.

(a) Given points P on AB, and Q on AC, find R on BC so that the perimeter of the triangle PQR is minimal.

(b) Given a point P' on AB, find points Q' on AC, and R' on BC so that the perimeter of the triangle $P'Q'R'$ is minimal.

(c) Find points P'' on AB, and Q'' on AC, and R'' on BC so that the perimeter of the triangle $P''Q''R''$ is minimal.

PROBLEM 8: Express an arbitrary positive integer n as the 2^{n-1} ordered sums of positive integers. For example, if $n = 4$, the 8 ordered sums are listed in the left column below:

4	2	$(= 2)$
$3+1$	2×1	$(= 2)$
$1+3$	1×2	$(= 2)$
$2+2$	3×3	$(= 9)$
$2+1+1$	$3\times 1\times 1$	$(= 3)$
$1+2+1$	$1\times 3\times 1$	$(= 3)$
$1+1+2$	$1\times 1\times 3$	$(= 3)$
$1+1+1+1$	$1\times 1\times 1\times 1$	$(= 1)$

The entries in the right column are obtained from the corresponding ones in the left column by

(a) Changing all additions to multiplications;

(b) Changing all integers $k \geq 3$ to 2;

(c) Changing 2 to 3;

(d) Keeping 1 unchanged.

Finally, add all the products in the right column. For $n = 4$, we obtain

$$2+2+2+9+3+3+3+1 = 25\ (= 5^2).$$

Prove or disprove: For every positive integer n, the sum of all the products in the right column is always a perfect square.