UNM-PNM STATEWIDE MATHEMATICS CONTEST XXXVII

FEBRUARY 5th, 2005 SECOND ROUND SOLUTIONS

**PROBLEM 1:** *Abran, Alisa, Ava and Alejandro are walking home in the middle of the night. It is very dark and they only have one lantern. They have to cross a wooden bridge. The bridge is in very poor condition and can support at most two of them at a time. Those crossing the bridge need the lantern so as not to fall through the cracks in the old wood, therefore the lantern needs to be transported back and forth until all of them have crossed the bridge.*

*Alisa is very fast and can cross the bridge in* 1 *minute. Abran is also quite fast and can cross it in* 2 *minutes. Ava is less fast but can still do it in* 5 *minutes. Finally Alejandro, who is very scared of heights and of the darkness, needs at least* 10 *minutes to cross the bridge.*

*We know the bridge will collapse after* 18 *minutes of walking over it. Will all of them be able to cross the bridge safely? If YES, describe how, and in how many minutes they can cross. If NOT, explain why and tell us what is the least amount of minutes in which all four friends can cross safely.*

**ANSWER:** YES, they can cross safely in 17 minutes.

**SOLUTION:** One thinks about this problem and often the first idea is to have Alisa, the fastest kid, as the lantern bearer. In this scenario Alisa will carry the lantern back and forth until everybody has crossed. Hence she will cross the bridge 3 times with each of the slower children (2, 5, and 10 minutes each crossing), and she will return twice on her own (1 minute each crossing). That requires a total of $2+5+10+1+1=19$ minutes .... One realizes that not all of them will cross safely, the bridge will collapse before the last two children (Alisa and somebody else) finish crossing. Many of you thought this was the solution, not a happy ending.

However some of you realized (or knew already the problem) that one will save time if the slowest children (Ava and Alejandro) will cross together. However if one of them has to return with the lantern whatever was saved in time will be wasted. So we have to ensure that once Ava and Alejandro cross together they do not have to return. We can achieve this by sending first the fastest children (Alisa and Abran) (2 minutes), sending back either of them (say Abran) (2 minutes), then sending the slowest (Ava and Alejandro) (10 minutes), now Alisa which is on the safe side can return with the lantern (1 minute) and cross with Abran (2 minutes). After $2+2+10+1+2=17$ minutes all the children have crossed safely the bridge. And there is a happy ending after all (provided they figure out the right strategy in less than one minute ;-).

**PROBLEM 2:***Let $x, y, z$ be real numbers. Suppose $(x+1)(y+1)(z+1) \neq 0$, and*

$$\frac{x}{x+1} + \frac{y}{y+1} + \frac{z}{z+1} = 1. \tag{1}$$

*Find all possible values of the quantity $(2xyz + xy + yz + zx)$ for all $x, y, z$ with the above properties.*

**ANSWER:** The only possible value of $(2xyz + xy + yz + zx)$ is 1.

**SOLUTION:** One could try some experiments. This entails finding triples $(x, y, z)$ different than $-1$, so that $(x+1)(y+1)(z+1) \neq 0$, and such that (1) holds.

Here are some examples of such triples,

- If $x = y = z$ then $\frac{3x}{x+1} = 1$, that is $3x = x+1$, or $x = y = z = 1/2$. Substituting into the desired expression we obtain $2(1/2)^3 + 3(1/2)^2 = 2/8 + 3/4 = 1/4 + 3/4 = 1$.
- If $x = 0$ and $y = z$ then $\frac{2y}{y+1} = 1$, that is $2y = y+1$, hence $y = z = 1$. This time the expression is $2 \times 0 \times 1^2 + 0 \times 1 + 1^2 + 1 \times 0 = 1$.

  By symmetry same will happen in the cases $y = 0$, $x = z = 1$, and $z = 0$, $x = y = 1$.

From these results one could GUESS that there is just one possible value for the expression, and that such value must be 1. However this is nothing more than a guess since there are infinitely many triples $(x, y, z)$ with the desired properties (convince yourselves that that is indeed the case).

A proof is hence required and there is not much to do except for doing some algebra. Multiply both sides of equation (1) by $(x+1)(y+1)(z+1)$, the non-zero common denominator, and simplify,

$$\begin{aligned} x(y+1)(z+1) + y(x+1)(z+1) + z(x+1)(y+1) &= (x+1)(y+1)(z+1) \\ 3xyz + 2xy + 2yz + 2zx + x + y + z &= xyz + xy + yz + zx + x + y + z + 1 \end{aligned}$$

Now collect and cancel similar terms, to obtain

$$2xyz + xy + yz + zx = 1.$$

**PROBLEM 3:** *Amy has the following rule to distribute candies on Halloween: the first child to come receives a 23rd of the candies plus one candy, the second one receives a 23rd of the remaining candies plus two candies, the third one receives a 23rd of the remaining candies plus three candies, etc. All the candies were given away and all the kids received the same amount of candies. How many children visited Amy? How many candies did each child get?*

**ANSWER:** There are 3 possible solutions:

1. The solution we expected was: 22 children visited Amy, and she had 506 candies. Each child received 23 candies.
2. Zero children visited Amy, and she had zero candies to distribute.
3. One child visited Amy, and Amy had $23/22$ candies. The lonely child received $(23/22)/23 + 1$ candies, that is all Amy had.

**SOLUTION:** Let $X$ = total number of candies, $N$ = number of children visiting Amy, and let $A$ = number of candies received by each child. According to the rules, the first child received a 23rd of the total number of candies $X$ plus one candy,

$$A = \frac{X}{23} + 1.$$

The second child received a 23rd of the total number of candies left, $X - A$, plus two candies,

$$A = \frac{X - A}{23} + 2.$$

The $k$th child received a 23rd of the total number of candies left, $X - (k-1)A$, plus $k$ candies,

$$A = \frac{X - (k-1)A}{23} + k.$$

The last child ($N$th child) received a 23rd of the total number of candies left, $X - (N-1)A$, plus $N$ candies,

$$A = \frac{X - (N-1)A}{23} + N.$$

At this point, all candies were gone, that is $N \times A = X$. We have a set of $N$ linear equations in the variables $X, A$. Given any two of them they will have a unique solution. Note that we are assuming that at least 2 children visited Amy. Substitute $A = \frac{X}{23} + 1$ given by the first equation into the second equation to get a linear equation in $X$,

$$\frac{X}{23} + 1 = \frac{X - (\frac{X}{23} + 1)}{23} + 2$$

Solve for $X$,

$$\begin{aligned} X + 23 &= X - \left(\frac{X}{23} + 1\right) + 2 \times 23 \\ X &= \frac{22}{23}X + 22 \\ 23X &= 22X + 22 \times 23 \\ X &= 506. \end{aligned}$$

Given $X = 506$ we can find $A = \frac{X}{23} + 1 = \frac{506}{23} + 1 = 22 + 1 = 23$. Given $X$ and $A$ we can find $N = X/A = 506/23 = 22$. Hence if *at least* 2 children visited Amy (otherwise we could not have started our argument), then 22 children must have visited her and each received 23 candies. One might wonder if all other $N - 2$ equations are satisfied. It suffices to check that the generic $k$th equation is satisfied for this choice of values of $A$ and $X$, more precisely,

$$\frac{X - (k-1)A}{23} + k = \frac{506 - (k-1)23}{23} + k = 22 - (k-1) + k = 22 + 1 = 23 = A,$$

and we are done!!! Some of you wrote a more or less complete table checking each of the 22 equations:

| Kid | Candies handed out | Candies left |
|---|---|---|
| 1 | $\frac{506}{23} + 1 = 22 + 1 = 23$ | $506 - 23 = 483$ |
| 2 | $\frac{483}{23} + 2 = 21 + 2 = 23$ | $483 - 23 = 460$ |
| 3 | $\frac{460}{23} + 3 = 20 + 3 = 23$ | $460 - 23 = 437$ |
| 4 | $\frac{437}{23} + 4 = 19 + 4 = 23$ | $437 - 23 = 414$ |
| $\vdots$ | $\vdots \;=\; \vdots \;=\; \vdots$ | $\vdots \;=\; \vdots$ |
| 20 | $\frac{69}{23} + 20 = 3 + 20 = 23$ | $69 - 23 = 46$ |
| 21 | $\frac{46}{23} + 21 = 2 + 21 = 23$ | $46 - 23 = 23$ |
| 22 | $\frac{23}{23} + 22 = 1 + 22 = 23$ | $23 - 23 = 0$ |

The phrasing of the problem was such that it seemed that some children had come to Amy's house, at least three. However if we disect the wording, we did not said it explicitely, so we cannot rule out the possibility of no children showing up, or only one. We should work out both possibilities[1]

If $N = 0$ then no child visited Amy and then $X = N \times A = 0$, and since she ended up emptyhanded she had zero candy to begin with. The value of $A$ is irrelevant in this case... Two students noticed this solution: **10th grader Leandra Boucheron from El Dorado HS,** and **9th grader Punit Sha from Albuquerque Academy**.

If $N = 1$ then this time $A = X = \frac{X}{23} + 1$, solving for $X$ we obtain a non-integer solution $X = 23/22$, which we could have discarded had we explicitely asked for whole candies. We did not make that assumption, pressumably Amy could have cut the candies in smaller pieces. This solution was found by only one student: **9th grader Nathaniel Zakahi from Las Cruces HS**.

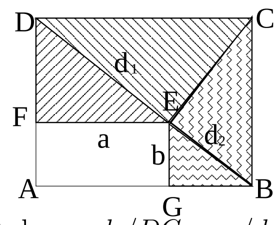
The one thing we will keep whole is the children, $N$ is a natural number!

**PROBLEM 4:** *Suppose $E$ is the foot of the perpendicular from $C$ to diagonal $BD$ in rectangle $ABCD$. If the lengths of perpendiculars from $E$ to $AD$ and $AB$ are $a$ and $b$, respectively, express the length $d$ of diagonal $BD$ in terms of $a$ and $b$.*

**ANSWER:** $d = BD = (a^{2/3} + b^{2/3})^{3/2}$.

[1] When composing the exam we did not think about these other possibilities. It was not until 9th grader Punit Sha from Albuquerque Academy contacted us to confirm the validity of the zero solution that we realized there was more than one solution.

**Solution 1 (inspired by the work of Tony Huan, 8th grader from Desert Ridge MS):**




Denote by $F$ and $G$ the foots of the perpendiculars from $E$ to $DA$ and $AB$ respectively. Denote by $d_1 = DE$, $d_2 = EB$, note that $FE = a$ and $EG = b$. We will find a formula for $d_1$ in terms of $a$ and $b$ and we will note that the same formula with the roles of $a$ and $b$ interchanged will work for $d_2$, finally $d = d_1 + d_2$.

There are many similar triangles, for example, $\triangle DEC \sim \triangle EFD$, hence $d_1/DC = a/d_1$, thus $d_1^2 = a \times DC$. On the other hand, $DC = a + GB$, and by Pythagoras theorem $GB = \sqrt{EB^2 - b^2}$. From $\triangle EBG \sim \triangle BCE$ we conclude that $EB^2 = b \times BC$. Notice also that $BC = b + FD$, and again by Pythagoras $FD = \sqrt{d_1^2 - a^2}$. Hence

$$\begin{aligned} EB^2 &= b \times BC = b(b + FD) \;=\; b^2 + b\sqrt{d_1^2 - a^2}, \\ GB &= \sqrt{EB^2 - b^2} \;=\; \sqrt{b\sqrt{d_1^2 - a^2}} \;=\; b^{1/2}(d_1^2 - a^2)^{1/4}, \\ DC &= a + GB \;=\; a + b^{1/2}(d_1^2 - a^2)^{1/4}. \end{aligned}$$

We conclude that

$$d_1^2 \;=\; a \times DC \;=\; a^2 + ab^{1/2}(d_1^2 - a^2)^{1/4}.$$

Bring $a^2$ to the left-hand-side, and notice that the quantity $d_1^2 - a^2$ appears on both sides, collect them together to get,

$$(d_1^2 - a^2)^{3/4} = ab^{1/2} \quad \Rightarrow \quad d_1 = \sqrt{a^2 + a^{4/3}b^{2/3}} = a^{2/3}\sqrt{a^{2/3} + b^{2/3}}.$$

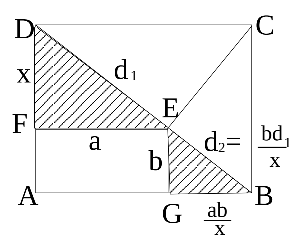
Similarly, $d_2 = b^{2/3}\sqrt{a^{2/3} + b^{2/3}}$. Lo and behold,

$$d = d_1 + d_2 = a^{2/3}\sqrt{a^{2/3} + b^{2/3}} + b^{2/3}\sqrt{a^{2/3} + b^{2/3}} = (a^{2/3} + b^{2/3})^{3/2},$$

or more symmetrically $d^{2/3} = a^{2/3} + b^{2/3}$.

Since the picture in the exam was ambiguous and a number of students thought the question was to write $d_1$ in terms of $a$ and $b$, we gave full credit for the correct computation of $d_1$.

**Solution 2 (inspired by the work of Kristin Cordwell, 8th grader from Jackson MS):** Let $F, G$ be as in the previous proof. We will use the Pythagorean theorem many times to find various lengths in the diagram. We will compute $d_1$, and a similar computation or a symmetry argument will work for $d_2$.




First, let $x = FD = \sqrt{d_1^2 - a^2}$. Next can compute $GB$ from $\triangle DEF \sim \triangle EBG$,

$$BE = \frac{bd_1}{x}, \qquad GB = \frac{ab}{x}.$$

Notice that $BC = b + x$, hence by Pythagoras,

$$CE^2 = BC^2 - BE^2 = (b + x)^2 - \frac{b^2 d_1^2}{x^2}.$$

Finally, notice that $DC = a + GB = a + \dfrac{ab}{x}$, and once more by Pythagoras,

$$d_1^2 \;=\; DC^2 - CE^2 \;=\; \left(a + \frac{ab}{x}\right)^2 - \left((b + x)^2 - \frac{b^2 d_1^2}{x^2}\right)$$

$$
\begin{aligned}
&= a^2 + \frac{2a^2b}{x} + \frac{a^2b^2}{x^2} - \left(b^2 + 2bx + x^2 - \frac{b^2d_1^2}{x^2}\right) \\
&= a^2 + \frac{2a^2bx + a^2b^2 - b^2x^2 - 2bx^3 - x^4 + b^2d_1^2}{x^2} \\
&= a^2 + \frac{2a^2bx + a^2b^2 + b^2(d_1^2 - x^2) - 2bx^3 - x^4}{x^2} \\
&= a^2 + \frac{2a^2bx + a^2b^2 + b^2a^2 - 2bx^3 - x^4}{x^2}
\end{aligned}
$$

In the last identity we used the fact that $a^2 = d_1^2 - x^2$. Subtracting $a^2$ on both sides, using now that $x^2 = d_1^2 - a^2$, and multiplying by $x^2$ both sides of the equation, gives as $x^4 = 2a^2bx + 2a^2b^2 - 2bx^3 - x^4$. Hence $x$ must be a solution to the quartic polynomial

$$2x^4 + 2bx^3 - 2a^2bx - 2a^2b^2 = 0,$$

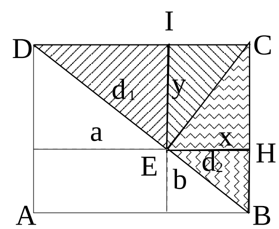
which can be factored easily into $2(x^3 - a^2b)(b + x) = 0$. There is only one positive solution to the equation, hence $x = a^{2/3}b^{1/3}$. We conclude that

$$d_1 = \sqrt{a^2 + x^2} = \sqrt{a^2 + a^{4/3}b^{2/3}} = a^{2/3}\sqrt{a^{2/3} + b^{2/3}}.$$

**Solution 3:** Denote by $H$ and $I$ the feet of the perpendicular lines dropped from $E$ onto sides $CD$ and $CB$ respectively. Let $x = EH$, $y = EI$.

Let $d_1 = DE$, $d_2 = EB$, hence $d = d_1 + d_2$. By Pythagoras,

$$d_1^2 = a^2 + y^2, \quad d_2^2 = b^2 + x^2, \quad \text{and} \quad d^2 = (x + a)^2 + (y + b)^2.$$




Hence,

$$
\begin{aligned}
d &= \sqrt{(x + a)^2 + (y + b)^2} \qquad (2) \\
d &= \sqrt{a^2 + y^2} + \sqrt{b^2 + x^2}, \qquad (3)
\end{aligned}
$$

In either case, if we can write $x$ and $y$ in terms of $a$ and $b$, then plugging the corresponding values into (3) or (2) would give us an expression for $d$ in terms of $a$ and $b$.

The following triangles are similar, $\triangle EBH$, $\triangle CEH$, $\triangle DEI$, $\triangle CEI$. From $\triangle EBH \sim \triangle CEH$, and $\triangle DEI \sim \triangle CEI$ we get that

$$\frac{x}{b} = \frac{y}{x}, \quad \text{and} \quad \frac{y}{a} = \frac{x}{y}.$$

A system of two equations on the variables $x$ and $y$. Solving the first one for $y$ we get $y = x^2/b$, plugging this into the second one we get,

$$\frac{x^4}{ab^2} = x \implies x^3 = ab^2.$$

We get expressions for $x$ and $y$ in terms of $a$ and $b$ as desired,

$$x = a^{1/3}b^{2/3}, \quad y = a^{2/3}b^{1/3}.$$

Substitute these formulae into (3) to obtain,

$$
\begin{aligned}
d &= \sqrt{(a + a^{1/3}b^{2/3})^2 + (b + b^{1/3}a^{2/3})^2} \\
&= \sqrt{a^{2/3}(a^{2/3} + b^{2/3})^2 + b^{2/3}(a^{2/3} + b^{2/3})^2} \\
&= \sqrt{(a^{2/3} + b^{2/3})(a^{2/3} + b^{2/3})^2} \\
&= (a^{2/3} + b^{2/3})^{3/2}.
\end{aligned}
$$

or into (2) to obtain,

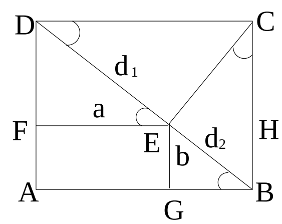
$$\begin{aligned} d &= \sqrt{a^2 + a^{4/3}b^{2/3}} + \sqrt{b^2 + b^{4/3}a^{2/3}} \\ &= a^{2/3}\sqrt{a^{2/3} + b^{2/3}} + b^{2/3}\sqrt{a^{2/3} + b^{2/3}} \\ &= (a^{2/3} + b^{2/3})^{3/2}. \end{aligned}$$

Any of the intermediate identities above give expressions for $d$ in terms of $a$ and $b$ as requested.

**Solution 4 (Prof. L.-S. Hahn):** Let $F$ and $G$, be the feet of the perpendicular dropped from $E$ onto sides $DA$ and $AB$ respectively as in the previous solutions.

Denote by $\theta$ the angle $\angle DBA$, note that

$$\theta = \angle DBA = \angle DEF = \angle ECB = \angle EDC.$$




Also note that,

$$d_1 = a \sec\theta, \quad \text{and} \quad d_2 = b \csc\theta.$$

From $\triangle CDE$, we get

$$CE = d_1 \tan\angle CDE = d_1 \tan\theta.$$

From $\triangle BCE$, we get

$$CE = d_2 \cot\angle BCE = d_2 \cot\theta.$$

Therefore,

$$\begin{aligned} d_1 \tan\theta &= d_2 \cot\theta, \\ \text{i.e.,} \qquad a \sec\theta \tan\theta &= b \csc\theta \cot\theta \\ (\tan\theta)^3 &= b/a. \end{aligned}$$

We have found $\tan\theta$ in terms of $a, b$. In turn we can write $\sec\theta$ and $\csc\theta$ in terms of $a, b$, and hence we can write $d$ in terms of $a, b$. More precisely,

$$\begin{aligned} d = d_1 + d_2 &= a\sec\theta + b\csc\theta \\ &= a\sqrt{1 + (\tan\theta)^2} + b\sqrt{1 + (\cot\theta)^2} \\ &= a\sqrt{1 + (b/a)^{2/3}} + b\sqrt{1 + (a/b)^{2/3}} \\ &= (a^{2/3} + b^{2/3})^{3/2}. \end{aligned}$$

**PROBLEM 5:** *Remember that* $\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$.

*For example* $\sum_{k=1}^{7} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$, *in this case* $n = 7$ *and* $a_k = k^2$.

**(a)** *Evaluate* $\displaystyle\sum_{k=1}^{5} \frac{1}{k(k+1)(k+2)}$.

**(b)** *Find an integer* $m$ *in terms of* $n$ *such that* $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{4} - \frac{1}{m}$.

**ANSWER: (a)** $\displaystyle\sum_{k=1}^{5} \frac{1}{k(k+1)(k+2)} = \frac{5}{21}$. **(b)** $m = 2(n+1)(n+2)$.

**SOLUTION: (a)** Computing directly, we get,

$$\begin{aligned}
\sum_{k=1}^{5} \frac{1}{k(k+1)(k+2)} &= \frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\frac{1}{3\times4\times5}+\frac{1}{4\times5\times6}+\frac{1}{5\times6\times7}\\
&= \frac{4\times5\times7+5\times7+2\times7+7+4}{4\times5\times6\times7}\\
&= \frac{140+35+14+7+4}{840} = \frac{200}{840} = \frac{5}{21}.
\end{aligned}$$

We are testing your understanding of the summation notation and your arithmetic abilities.

**(b)** A direct computation will not work this time, we must find a better way. Those of you who attended Dunham's talk might have been inspired by his example summing up the reciprocals of the triangular numbers! We will present three arguments, all based in different partial fraction decompositions

**Solution 1:** We can find numbers $A$, $B$, and $C$ such that

$$\frac{1}{k(k+1)(k+2)} = \frac{A}{k}+\frac{B}{k+1}+\frac{C}{k+2},$$

by arguments similar to those discussed in Problem 2 in the First Round Exam. In this case $A = C = 1/2$, $B = -1$. Therefore,

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} &= \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{2}{k+1}+\frac{1}{k+2}\right)\\
&= \frac{1}{2}\left[\left(\frac{1}{1}-\frac{2}{2}+\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{2}{4}+\frac{1}{5}\right)+\cdots\right.\\
&\qquad \cdots+\left(\cdots+\frac{1}{k+1}\right)+\left(\frac{1}{k}-\frac{2}{k+1}+\frac{1}{k+2}\right)+\left(\frac{1}{k+1}-\cdots\right)+\cdots\\
&\qquad \left.\cdots+\left(\frac{1}{n-1}-\frac{2}{n}+\frac{1}{n+1}\right)+\left(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right)\right].
\end{aligned}$$

Notice that the central negative terms cancel out most of the time with one term on the left and another on the right, except on the edges where one of them is missing. This is an example of a double *telescoping sum.* Cancelling everything there is to be cancelled, we are left with,

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} &= \frac{1}{2}\left[1-\frac{1}{2}-\frac{1}{n+1}+\frac{1}{n+2}\right]\\
&= \frac{1}{2}\left[\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right] = \frac{1}{4}-\frac{1}{2(n+1)(n+2)}.
\end{aligned}$$

We conclude that $m = 2(n+1)(n+2)$.

**Solution 2 (by 11th grader Lu Yang from United World College):** Denote by $S$ the sum we are trying to find,

$$S = \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)}.$$

Notice that the following partial fraction decompositions hold:

$$\frac{1}{k(k+1)} - \frac{1}{k(k+2)} = \frac{1}{k(k+1)(k+2)} = \frac{1}{k(k+2)} - \frac{1}{(k+1)(k+2)}.$$

Using the first partial fraction decomposition we obtain

$$S = \sum_{k=1}^{n} \left( \frac{1}{k(k+1)} - \frac{1}{k(k+2)} \right) = \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{6} - \frac{1}{8} \right) + \cdots + \left( \cdots - \frac{1}{k(k+2)} \right) + \left( \frac{1}{(k+1)(k+2)} - \cdots \right) +$$

$$\cdots + \left( \frac{1}{(n-1)n} - \frac{1}{(n-1)(n+1)} \right) + \left( \frac{1}{n(n+1)} - \frac{1}{n(n+2)} \right).$$

Shifting the parenthesis one term to the right gives us,

$$S = \frac{1}{2} - \left( \frac{1}{3} - \frac{1}{6} \right) - \cdots - \left( \frac{1}{k(k+2)} - \frac{1}{(k+1)(k+2)} \right) - \cdots - \left( \frac{1}{(n-1)(n+1)} - \frac{1}{n(n+1)} \right) - \frac{1}{n(n+2)}.$$

We can now use the second partial fraction decomposition to note that

$$S = \frac{1}{2} - \left( \frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \cdots + \frac{1}{(n-1) \times n \times (n+1)} \right) - \frac{1}{n(n+2)}.$$

The sum in parenthesis that is being subtracted in the middle corresponds to the initial sum up to $n-1$, that is, it equals $S - \frac{1}{n(n+1)(n+2)}$, hence

$$S = \frac{1}{2} - \left( S - \frac{1}{n(n+1)(n+2)} \right) - \frac{1}{n(n+2)} = \frac{1}{2} - S + \frac{1}{n(n+1)(n+2)} - \frac{1}{n(n+2)}.$$

Solving for $S$, we obtain,

$$2S = \frac{1}{2} - \frac{n}{n(n+1)(n+2)}, \quad \text{or} \quad S = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

We conclude that $m = 2(n+1)(n+2)$.

**Solution 3 (several students including 12th grader Robert Cordwell form Manzano HS):** This we consider the most efficient solution, it combines elements from the previous two solutions: partial fraction into only two terms which provide a *telescopic sum* of the simplest type, namely a sum of the form

$$\sum_{k=1}^{n} (a_k - a_{k+1}) = (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n-1} - a_n) + (a_n - a_{n+1}) = a_1 - a_{n+1}.$$

There is a third partial fraction decomposition into two terms, namely,

$$\frac{1}{k(k+1)(k+2)} = \frac{1/2}{k(k+1)} - \frac{1/2}{(k+1)(k+2)}.$$

Hence our sum $S$ equals a telescopic sum with term $a_k = \frac{1}{2k(k+1)}$

$$S = \sum_{k=1}^{n} \left( \frac{1}{2k(k+1)} - \frac{1}{2(k+1)(k+2)} \right) = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

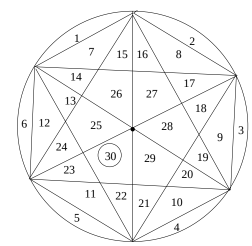
As before we conclude that $m = 2(n+1)(n+2)$.

**PROBLEM 6: (a)** *Given 6 points on a circle, how many chords are there having two of these 6 points as endpoints? What is the maximum possible number of intersections these chords can make in the interior of the circle? What is the maximum possible number of regions these chords can divide the interior of the circle?*

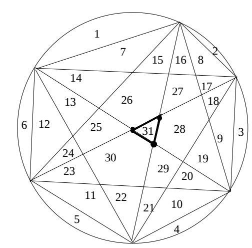
**(b)** *Given 12 points on the circle, how many chords are there having two of these 12 points as endpoints? What is the maximum possible number of intersections these chords can make in the interior of the circle? What is the maximum possible number of regions these chords can divide the interior of the circle?*

**ANSWER: (a)** 15 chords, 15 intersection points, 31 regions.
**(b)** 66 chords, 495 intersection points, 562 regions.

**SOLUTION:**




**(a)** This one can be done by hand. If we, like most people, draw a symmetric picture, then we will be misslead. Counting chords, points and regions in the case of the 6 points being the vertices of a regular hexagon, we will get the right number of chords, which is $15 = 5 + 4 + 3 + 2 + 1$ (5 chords from the first point, 4 new chords from the second point, 3 new chords from the third point, 2 new chords from the fourth chord, and 1 new chord from the fifth point, the sixth point does not contribute any new chord). However we will count only 13 interior intersection points and only 30 regions. See figure on the right, here the numbers correspond to the regions.



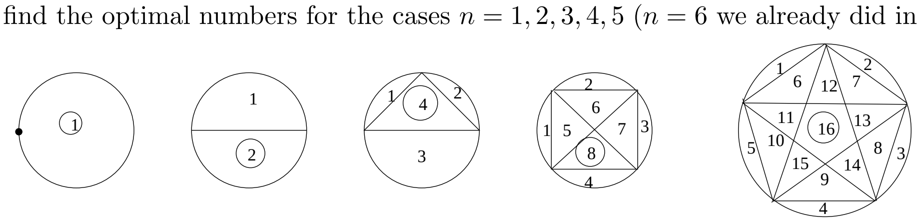

Notice that all intersection points except for the center occur as intersection of *only two chords*. The center is an intersection point for 3 chords (three diameters), and this is not the most efficient since we are seeking for the *maximum* number of intersection points. It should be clear that by moving slightly just one point then magically 3 intersection points appear where there was only one, and a new region is created where before we had the center of the circle. This time the picture is optimal since all intersection points are now obtained from the intersection of just two chords, and all chords that can intersect, are intersecting. We count 15 intersection points $(13 + 2)$, and 31 regions $(30 + 1)$.

**(b)** In the case of 12 points, a priori one might think that it can also be done by hand, but it is quite cumbersome. A more general method is worth finding.

Denote by $C_n$ = number of chords determined by $n$ points on the circle, $I_n$ = maximum number of intersection points determined by the chords inside the circle, and $R_n$ = maximum number of regions determined by the chords inside the circle.

**Experimental results, observing patterns and guessing formulas:**

Let us find the optimal numbers for the cases $n = 1, 2, 3, 4, 5$ ($n = 6$ we already did in part (a)),




Let us record the results on a table

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $C_n$ | 0 | 1 | 3 | 6 | 10 | 15 |
| $I_n$ | 0 | 0 | 0 | 1 | 5 | 15 |
| $R_n$ | 1 | 2 | 4 | 8 | 16 | 31 |

The table seems to have a clear pattern for the number of chords. To go from $C_n$ to $C_{n+1}$ we are adding $n$ (at least for $n \leq 6$), that is the following recurrence formula seems to hold,

$$C_{n+1} = C_n + n. \tag{4}$$

If we believe (4) holds for all $n$, we can use it to fill in the values of $C_n$ for $n \leq 12$,

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_n$ | 0 | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 45 | 55 | 66 |

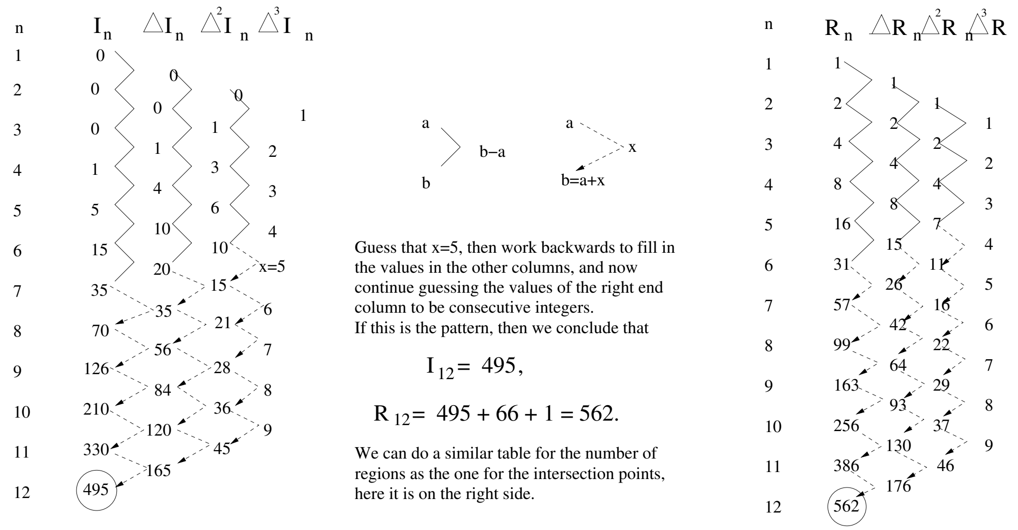
Hence, if this pattern holds then $C_{12} = 66$.

Notice that had we done the table up to $n = 5$ to guess the result for $n = 6$ we could have been mislead to conclude that $R_6 = 32 = 2^5$, and that $R_n = 2^{n-1}$ (in particular that $R_{12} = 2^{11}$) This is NOT the case! However with the correct table (up to $n = 6$) in front of us we might notice another pattern, for $n = 1, 2, 3, 4, 5, 6$ the following holds,

$$R_n = C_n + I_n + 1. \tag{5}$$

It turns out that formulae (4) and (5) are both true, but we have to prove them, so far they are nothing more than educated guesses. Furthermore even if we believe that (5) is true, we still need a way to figure out $I_n$ to use it.

To discover a patern for $I_n$ we could do what some of you did, which is to analize first differences, then second differences, and even third and fourth differences until the pattern of the differences is linear or constant. This argument was very well explained by **10th grader Zeev Friedman from La Cueva HS**, he also used it to guess the formula for Problem 5(b).




Some of you did exactly this and discovered the right answers, because these were indeed the correct paterns. Some of you even went farther and knew how to deduce formulas in terms of $n$ if the pattern were to hold. In the case of $I_n$, the fact that we get a linear pattern in the third differences $\Delta^3 I$ and we believe the pattern continues forever, means that the formula the original quantity $I_n$ obeys is a quartic polynomial in $n$[2], that is,

$$I_n = An^4 + Bn^3 + Cn^2 + Dn + E.$$

To discover the coefficients it will suffice to evaluate the polynomial at five known points, for example, $n = 1, 2, 3, 4, 5$ to obtain a system of 5 equations in the 5 unknowns $A, B, C, D, E$.

$$(I_1 =) \quad 0 \quad = \quad A + B + C + D + E,$$

[2] In general, if the linear pattern appears for the $k$-th differences $\Delta^k I$ (or equivalently, a constant pattern appears for the $(k+1)$-th differences) then the original quantity obeys a $k+1$ degree polynomial equation in $n$.

$$
\begin{aligned}
(I_2 =) \quad 0 &= 16A + 8B + 4C + 2D + E, \\
(I_3 =) \quad 0 &= 81A + 27B + 9C + 3D + E, \\
(I_4 =) \quad 1 &= 256A + 64B + 16C + 4D + E, \\
(I_5 =) \quad 5 &= 625A + 125B + 25C + 5D + E.
\end{aligned}
$$

This system can be solved, and the unique solution is,

$$A = \frac{1}{24}, \quad B = -\frac{1}{4}, \quad C = \frac{11}{24}, \quad D = -\frac{1}{4}, \quad E = 0.$$

If the pattern is to hold, then the formula for the number of intersection points is given by

$$I_n = \frac{n^4 - 6n^3 + 11n^2 - 6n}{24} = \frac{n(n-1)(n-2)(n-3)}{24}. \tag{6}$$

With this formula and formula (4) for the number of chords, plus formula (5) for the regions we can also guess a formula for the regions in terms of $n$,

$$R_n = \frac{n(n-1)(n-2)(n-3)}{24} + \frac{n(n-1)}{2} + 1. \tag{7}$$

Check that if we use the same method used for $I_n$ to find the coefficients of the quartic polynomial that fits the pattern table for $R_n$, the answer will coincide with (7).

We still need to justify formulas (4), (5) and (6). If they are valid, then they will imply (7).

**Verifying that our guesses hold:**

The most elegant and efficient way for proving (4) and (6) is recorded in Solution 2 below, but it requires working knowledge of combinatorial numbers. Here we present alternative proofs which are somehow convoluted, and requires knowledge of certain sums, namely:

$$\sum_{k=1}^{n} 1 = 1 + 1 + \cdots + 1 + 1 = n, \tag{8}$$

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}, \tag{9}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{10}$$

The last identity can be verified by mathematical induction, for example.

$C_n$ can be computed by a similar method as in the first part, namely, the first point contributes $(n-1)$ chords, the second $(n-2)$ new chords, ..., the $(n-2)$-th point contributes 2 new chords, the $(n-1)$-th point contributes 1 new chord, finally the $n$-th point contributes zero new chords, hence

$$C_n = (n-1) + (n-2) + \cdots + 2 + 1 = \frac{(n-1)n}{2},$$

where the last identity is an application of (9). Then our recurrence equation (4) clearly holds,

$$C_n = (n-1) + (n-2) + \cdots + 2 + 1 = (n-1) + C_{n-1}.$$

We will show that (6) holds by mathematical induction. It holds for $n = 1, 2, 3, 4, 5, 6$, assume it holds for $n$, we want to show it holds for $(n+1)$, namely,

$$I_{n+1} = \frac{(n+1)n(n-1)(n-2)}{24}.$$

Notice that

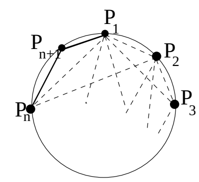
$$I_{n+1} = I_n + (I_{n+1} - I_n).$$

Let $\Delta I_n = I_{n+1} - I_n$, then by induction hypothesis,

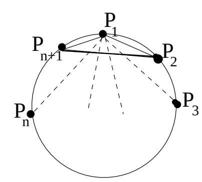
$$I_{n+1} = \frac{n(n-1)(n-2)(n-3)}{24} + \Delta I_n. \tag{11}$$

We need to compute $\Delta In$. We are trying to maximize the number of interior intersection points of the chords determined by $(n+1)$ points on the circle. Suppose the points are labeled clockwise $P_1, P_2, P_3, \ldots, P_{n+1}$. If we remove the last point $P_{n+1}$, and the $n$ chords it determines, we should have maximum number $I_n$ of interior intersection points determined by the $C_n$ chords determined by the first $n$ points, $P_1, \ldots, P_n$. We wish to count the maximum number of new intersection points $\Delta I_n$ created when $P_{n+1}$ is added.

- Chord $P_{n+1}P_1$ adds no interior intersection point, since $P_1$ is adjacent to $P_{n+1}$.
- Chord $P_{n+1}P_2$ adds as many intersection points as chords originating from $P_1$ there are, excluding the chords $P_{n+1}P_1$, and $P_2P_1$. There are $(n-2)$ such chords, hence $P_{n+1}P_2$ contributes $1 \times (n-2)$ new intersection points.

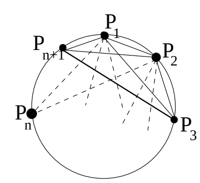



- Chord $P_{n+1}P_3$ adds as many intersection points as chords originating from $P_1$ and $P_2$ there are, excluding $P_1P_2$ and all those chords that have $P_{n+1}$ or $P_3$ as the other endpoint.
  - The excluded chords for $P_1$ are $P_1P_{n+1}$, $P_1P_2$, $P_1P_3$. The remaining $(n-3)$ chords originating from $P_1$ contribute each one a new intersection point.




  - The excluded chords for $P_2$ are $P_2P_{n+1}$, $P_1P_2$, $P_2P_3$. The remaining $(n-3)$ chords originating from $P_2$ contribute each one a new intersection point.

  All together, chord $P_{n+1}P_3$ contributes $2 \times (n-3)$ intersection points.




- Chord $P_{n+1}P_k$ contributes as many intersection points as chords originating from $P_1, P_2, \ldots, P_{k-1}$ there are , excluding those chords that have the other endpoint equal to $P_1, P_2, \ldots, P_{k-1}$, or $P_k$, or $P_{n+1}$.
  - The excluded chords for $P_1$ are $P_1P_{n+1}$, $P_1P_2$, $P_1P_3$,..., $P_1P_k$. The remaining $(n-k)$ chords originating from $P_1$ contribute each one a new intersection point. Hence point $P_1$ contributes $(n-k)$ new intersection points. Similarly for each of the other points $P_2$, $P_3$,..., $P_{k-1}$.

  All together, chord $P_{n+1}P_k$ contributes $(k-1) \times (n-k)$ intersection points.
- Notice that the last chord, $P_{n+1}P_n$ contributes zero intersection points, since the endpoints are adjacent points. Formula still holds in this case, set $k = n$, then $(n-1)(n-n) = 0$.

Adding up all new contributions we get that

$$\Delta I_n = 1 \times (n-2) + 2 \times (n-3) + \cdots + (k-1) \times (n-k) + \cdots + (n-2) \times 1. \tag{12}$$

Using the summation notation and its linear properties[3], we get that,

$$\begin{aligned}\Delta I_n &= \sum_{k=1}^{n}(k-1)(n-k) &(13)\\ &= \sum_{k=1}^{n}[(n+1)k-n-k^2] \;=\; (n+1)\sum_{k=1}^{n}k - n\sum_{k=1}^{n}1-\sum_{k=1}^{n}k^2. &(14)\end{aligned}$$

Evaluating the sums on the right hand side by (9), (8), and (10), we get,

$$\begin{aligned}\Delta I_n &= (n+1)\frac{n(n+1)}{2}-n(n)-\frac{n(n+1)(2n+1)}{6}\\ &= \frac{n\big[3(n+1)^2-6n-(n+1)(2n+1)\big]}{6} \;=\; \frac{n(n^2-3n+2)}{6}\\ &= \frac{n(n-1)(n-2)}{6}.\end{aligned}$$

We are ready to insert this into (11) to get,

$$\begin{aligned}I_{n+1} &= \frac{n(n-1)(n-2)(n-3)}{24}+\frac{n(n-1)(n-2)}{6}\\ &= \frac{n(n-1)(n-2)\big[(n-3)+4\big]}{24} \;=\; \frac{(n+1)n(n-1)(n-2)}{24}.\end{aligned}$$

Which is exactly what we wanted to prove. Formula (6) has been verified.

We can get a different formula for $I_n$, in terms of $\Delta I_m$ for $1 \leq m \leq n$,

$$I_n = (I_n - I_{n-1}) + (I_{n-1} - I_{n-2}) + \cdots + (I_3 - I_2) + I_2 = \Delta I_n + \Delta I_{n-1} + \Delta I_{n-2} + \cdots + \Delta I_3 + I_2.$$

This is yet another example of a telescoping sum. Remember that $I_2 = 0$, and using summation notation, we get

$$I_n = \sum_{m=3}^{n}\Delta I_m.$$

Substituting the formula for $\Delta I_m$ given by (12) we can write a very long formula for $I_n$ in terms of $n$, which can be compactified by using the summation notation once more as in (13),

$$I_n = \sum_{m=3}^{n}\left(\sum_{k=1}^{m}(k-1)(m-k)\right). \tag{15}$$

We can use this formula to compute $I_6$ and $I_{12}$ if we are pacient enough to carry on the calculations.

As for formula (5) we will also prove it by induction. It holds for $n = 1, 2, 3, 4, 5$, assume is true for $n$, show it holds for $(n+1)$, that is,

$$R_{n+1} = C_{n+1} + I_{n+1} + 1.$$

Notice that given a partition of a circle by $N$ chords into $\mathcal{R}_N$ regions if we add one more chord, then the number of regions increments by one as we start travelling on the chord starting at one endpoint until we reach the first intersection point if there is at least one, or the other endpoint if there is no intersection point ( we are subdividing an existing region into two regions). Then a second region is subdivided into two regions until we hit the second intersection point, we keep on doing this until we reach the other endpoint. The

[3]Namely: $\displaystyle\sum_{k=1}^{n}(Ca_k + Db_k) = C\sum_{k=1}^{n}a_k + D\sum_{k=1}^{n}b_k.$

number of regions have incremented exactly by the number $\mathcal{I}_{N+1}$ of interior intersection points introduced by the $N+1$ chord plus one. That is

$$\mathcal{R}_{N+1} = \mathcal{R}_N + \mathcal{I}_{N+1} + 1.$$

In our case, $n$ points determine $N = C_n$ chords, and $R_n = \mathcal{R}_{C_n}$ regions. Adding an extra point, amounts to adding $n$ new chords, and each chord will increase the number of regions by the number of intersection points it determines plus 1. All together we conclude that,

$$R_{n+1} = R_n + \Delta I_n + n.$$

We can now use the induction hypothesis (5), to get

$$R_{n+1} = C_n + I_n + 1 + \Delta I_n + n.$$

Remember now that $C_{n+1} = C_n + n$, and $I_{n+1} = I_n + \Delta I_n$, we get then

$$R_{n+1} = C_{n+1} + I_{n+1} + 1.$$

Which is what we wanted to prove.

Not many students gave complete proofs of these facts, among them I would like to mention **9th grader Benjamin Dozier from Los Alamos HS**.

**Solution 2 (Prof. L.-S. Hahn):** Notice that each chord is uniquely determined by a pair of points, hence $C_n$ equals the number of ways we can choose two different points from the $n$ points where the order in which we choose the points does not matter (that is the chord determined by points $A$ and $B$ is the same as the one determined by points $B$ and $A$). For those of you familiar with combinatorial numbers that is exactly the quantity given by

$$C_n = \binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}.$$

Remember that $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$, and $\binom{n}{m} = \frac{n!}{m!(n-m)!}$, for $m \leq n$ (we say "n choose m") [4].

To maximize the number of intersection points, we would like each one of them to be given by the intersection of at most two chords. In this optimal configuration that we are seeking, each interior intersection point is determined by 2 different chords that do not share and end point. But, 2 different chords that do not share and end point are determined by four different points on the circle, and given four different points on the circle there is only one pair of chords that creates an interior intersection point. Therefore there is a one to one correspondence between interior intersection points and sets of four points chosen form the given $n$ points where the order in which we choose the points does not matter. For those of you familiar with combinatorial numbers that is exactly the quantity given by

$$I_n = \binom{n}{4} = \frac{n!}{4!(n-4)!} = \frac{n(n-1)(n-2)(n-3)}{4!}.$$

To find $R_n$ we use the same recurrence formula (5) discussed in the previous solution.

Lo and behold,

$$C_6 = \binom{6}{2} = \frac{6 \times 5}{2} = 15, \quad I_6 = \binom{6}{4} = \frac{6 \times 5 \times 4 \times 3}{24} = 15, \quad R_6 = 15 + 15 + 1 = 31;$$

[4] $\binom{n}{m}$ is the number of ways we can choose $m$ objects out of $n$ given ones, where the order in which we select them doesn't matter, they are the combinatorial numbers also denoted $C^n_m$ that appear in Pascal's triangle (see next page), which also appear as coefficients of the polynomial $(x+1)^n$.

$$C_{12} = \binom{12}{2} = \frac{12 \times 11}{2} = 66, \quad I_{12} = \binom{12}{4} = \frac{12 \times 11 \times 10 \times 9}{24} = 495, \quad R_{12} = 66 + 495 + 1 = 562.$$

Since we have obtained two different looking formulas for $I_n$, namely (6) and (15), as a bonus we obtain the following identity, which a priori is not obvious at all,

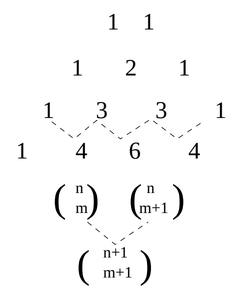
$$\binom{n}{4} = \sum_{m=3}^{n} \left( \sum_{k=1}^{m} (k-1)(m-k) \right).$$

Notice that we can also use this combinatorial ideas to compute $\Delta I_n$. The intersection points introduced by the $n+1$ point $P_{n+1}$, are in a one to one correspondence with sets of three different points chosen from the $n$ other points, and that is given by the combinatorial number "n choose 3", that is,

$$\Delta I_n = \binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}.$$

We can now get a basic formula for combinatorial numbers from the formula $I_{n+1} = I_n + \Delta I_n$, namely,

$$\binom{n+1}{4} = \binom{n}{4} + \binom{n}{3}.$$




Exercise: show that the following formula holds in general for $1 \leq m < n$,

$$\binom{n+1}{m+1} = \binom{n}{m+1} + \binom{n}{m}.$$

This is the basic formula in the construction of Pascal's triangle which encodes in its $n$-th row the $n$-th combinatorial numbers, or triangular numbers!

**PROBLEM 7:** *Let $ABC$ be an acute triangle. Recall that an acute triangle has all angles less than $90°$.*

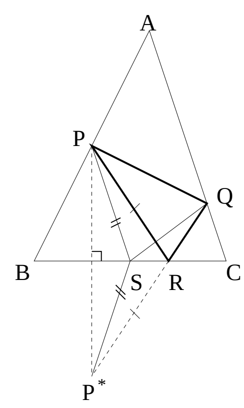
**(a)** *Given points $P$ on $AB$, and $Q$ on $AC$, find $R$ on $BC$ so that the perimeter of the triangle $PQR$ is minimal.*

**(b)** *Given a point $P'$ on $AB$, find points $Q'$ on $AC$, and $R'$ on $BC$ so that the perimeter of the triangle $P'Q'R'$ is minimal.*

**(c)** *Find points $P''$ on $AB$, and $Q''$ on $AC$, and $R''$ on $BC$ so that the perimeter of the triangle $P''Q''R''$ is minimal.*

**ANSWER:** See Solution.

**SOLUTION:** **(a)** The answer to this problem is the same as the answer to the billiard problem **8(a)** in the first round exam. The point $R$ on the side $BC$ that will minimize the perimeter of $\triangle PQR$ is the point we will have to aim at if we were hitting a ball at point $P$ and we would like it to bounce on side $BC$ and hit a ball at point $Q$.




Denote by $P^*$ the point symmetric to $P$ with respect to side $BC$. Let $R$ be the intersection point of $BC$ and $P^*Q$.

**Claim:** $\triangle PQR$ has minimal perimeter.

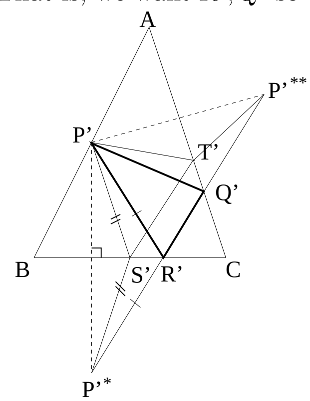
Note that minimizing the perimeter of $\triangle PQS$ for $S$ a point on $BC$ is the same as minimizing $PS + SQ$, since the side $PQ$ is fixed. By construction, $PR + RQ = P^*R + RQ = P^*Q$ (straight line). For any other point $S$ on $BC$, it is still true that $PS = P^*S$, however (see the picture),

$$PS + SQ = P^*S + SQ \geq P^*Q = PR + RQ.$$

Hence $R$ minimizes the desired quantity.

**(b)** We can still view this as a billiard problem with an acute angled corner $\angle BCA$. This time we want to find points $R'$ on $BC$ and $Q'$ on $AC$ so that when we the ball sitting on $P'$ is aimed at $R'$ it bounces towards side $AC$ and hits it exactly at $Q'$, and then bounces back towards $P'$. That is, we want $R', Q'$ so that $P'R'$ and $R'Q'$ are reflection trajectories, and so are $R'Q'$ and $Q'P'$.

Denote by $P'^*$ the point symmetric to $P'$ with respect to side $BC$, and by $P'^{**}$ the point symmetric to $P'$ with respect to side $AC$. Let $R'$ and $Q'$ be the intersection points of $P'^*P'^{**}$ with sides $BC$ and $AC$ respectively. Note that $P'R' = P'^*R'$ and $P'Q' = P'^{**}Q'$.




**Claim:** $\triangle P'Q'R'$ has minimal perimeter.

Notice that the perimeter of $\triangle P'R'Q'$ is

$$P'R' + R'Q' + Q'P' = P'^*P'^{**} \quad \text{(straight line)}.$$

Given any other points $S'$, $T'$ on $BC$ and $AC$ respectively, by construction it is still true that $P'S' = P'^*S'$ and $P'T' = T'P'^{**}$. Therefore, the perimeter of $\triangle P'S'T'$ is

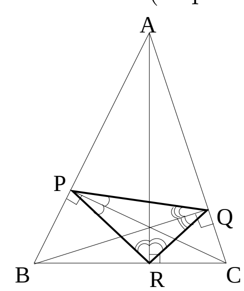
$$P'^*S' + S'T' + T'P'^{**} \geq P'^*P'^{**}.$$

Hence $\triangle P'Q'R'$ has minimal perimeter as claimed.

**(c)** This time we do not have an initial point, however once we have a candidate for $P$ or $Q$ or $R$ (we will drop the double primes for simplicity in the notation), by part (b) the other two points must obey the reflection properties. That is, if $\triangle ABC$ is a billiard table, then we are searching for points $P, Q, R$, so that if we aim a ball sitting on any of them at the other points, the trajectory of the ball will be the perimeter of $\triangle PQR$. If we draw the line through $P$ (respectively $R, Q$) perpendicular to $AB$ (respectively $BC$, $AC$), then it will bisect $\angle QPR$ (respectively $\angle PRQ$, $\angle RQP$).

This is a property of the *orthotriangle*[a], that is the triangle whose vertices are the feet $H_A$, $H_B$, $H_C$ of the perpendiculars dropped from $A$, $B$ and $C$ respectively (notice that $H_A$, $H_B$, $H_C$ lie on the sides of $\triangle ABC$ because it is assumed to be an acute triangle).

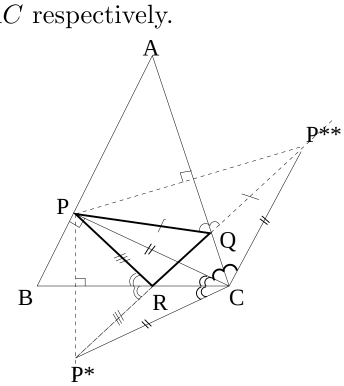
[a]Try to prove this fact. It will be a corollary of the proof below.




**Claim:** $P = H_C$, $Q = H_B$, $R = H_A$ are the points that minimize the perimeter of $\triangle PQR$.

**Proof of the Claim (by Prof. L.-S. Hahn):** Given point $P$ on side $AB$, denote by $P^*$ the point symmetric to $P$ with respect to side $BC$, and by $P^{**}$ the point symmetric to $P$ with respect to side $AC$. Let $R$ and $Q$ be the intersection points of $P^*P^{**}$ with sides $BC$ and $AC$ respectively.

Note that $PR = P^*R$ and $PQ = P^{**}Q$, and hence $\angle PQA = \angle P^{**}QA$ and $\angle PRB = \angle P^*RB$. These in turn imply that $\triangle PQC$ is not only similar but also congruent to $\triangle P^{**}QC$, since $\angle PQC = \angle P^{**}QC$. We conclude that $PC = P^{**}C$, and that $\angle PCQ = \angle P^{**}CQ$. The same argument shows that $\triangle PRC$ is congruent to $\triangle P^*RC$, hence, $PC = P^*C$, and $\angle PCR = \angle P^*CR$.
Therefore, recalling that $\angle ACB = \angle PCQ + \angle PCR$, we conclude that




$$\angle P^{**}CP^* = 2\angle ACB.$$

Since $PC = P^*C = P^{**}C$, then $\triangle P^{**}CP^*$ is ISOSCELES at vertex $C$, and the angle at vertex $C$ is INDEPENDENT of the choice of the point $P$.

Moreover, the side $P^*P^{**} = P^*R + RQ + QP^{**} = PR + RQ + QP$, is equal to the perimeter of $\triangle PQR$, which is the quantity we want to minimize. Because the angle at vertex $C$ of $\triangle P^{**}CP^*$ is independent of the choice of $P$, then the side $P^*P^{**}$ will be minimized whenever the length of sides $P^*C$ and $P^{**}C$ are minimized. Both sides have length equal to $PC$ and that length is minimized when $PC$ is perpendicular to $AB$, that is when $P = H_C$.

Similar arguments will prove that $R = H_A$ and $Q = H_B$.

We have shown that the orthotriangle $\triangle H_AH_BH_C$ minimizes the perimeter of all triangles inscribed on $\triangle ABC$. Furthermore we have shown that the orthotriangle has the reflection property at each vertex, that is, the heights $H_AA$, $H_BB$, $H_CC$ bisect the angles $\angle H_BH_AH_C$, $\angle H_CH_BH_A$, $\angle H_AH_CH_B$, respectively.

**PROBLEM 8:** *Express an arbitrary positive integer $n$ as the $2^{n-1}$ ordered sums of positive integers. For example, if $n = 4$, the 8 ordered sums are listed in the left column below:*

| | | |
|---|---|---|
| $4$ | $2$ | $(= 2)$ |
| $3+1$ | $2 \times 1$ | $(= 2)$ |
| $1+3$ | $1 \times 2$ | $(= 2)$ |
| $2+2$ | $3 \times 3$ | $(= 9)$ |
| $2+1+1$ | $3 \times 1 \times 1$ | $(= 3)$ |
| $1+2+1$ | $1 \times 3 \times 1$ | $(= 3)$ |
| $1+1+2$ | $1 \times 1 \times 3$ | $(= 3)$ |
| $1+1+1+1$ | $1 \times 1 \times 1 \times 1$ | $(= 1)$ |

*The entries in the right column are obtained from the corresponding ones in the left column by*

*(a) Changing all additions to multiplications;*

*(b) Changing all integers $k \geq 3$ to 2;*

*(c) Changing 2 to 3;*

*(d) Keeping 1 unchanged.*

*Finally, add all the products in the right column. For $n = 4$, we obtain*

$$2 + 2 + 2 + 9 + 3 + 3 + 3 + 1 = 25\ (= 5^2).$$

*Prove or disprove: For every positive integer $n$, the sum of all the products in the right column is always a perfect square.*

**ANSWER:** Yes, for every positive integer $n$, the sum of all the products in the right column is always a perfect square, in fact the square of a Fibonacci number.

**SOLUTION:** The first thing to do in a problem like this is to experiment with other values of $n$. Let us see what happens if we perform this crazy procedure for $n = 1, 2, 3$. Denote by $S_n$ the sum of the products on the left column for the table corresponding to $n$.

| | | |
|---|---|---|
| $1$ | $1$ | $(= 1)$ |

$S_1 = 1 = 1^2$

| | | |
|---|---|---|
| $2$ | $2$ | $(= 3)$ |
| $1+1$ | $1 \times 1$ | $(= 1)$ |

$S_2 = 3 + 1 = 4 = 2^2$

| | | |
|---|---|---|
| $3$ | $2$ | $(= 2)$ |
| $2+1$ | $3 \times 1$ | $(= 3)$ |
| $1+2$ | $1 \times 3$ | $(= 3)$ |
| $1+1+1$ | $1 \times 1 \times 1$ | $(= 1)$ |

$S_3 = 2 + 3 + 3 + 1 = 9 = 3^2$

We are given that $S_4 = 5^2$. It works so far, but before advancing a hypothesis, let us check one more case, $n = 5$. This time there will be $2^4 = 16$ different combinations,

| | | | | | |
|---|---|---|---|---|---|
| $5$ | $2$ | $(= 2)$ | $3+1+1$ | $2\times1\times1$ | $(= 2)$ |
| $4+1$ | $2\times1$ | $(= 2)$ | $1+3+1$ | $1\times2\times1$ | $(= 2)$ |
| $1+4$ | $1\times2$ | $(= 2)$ | $1+1+3$ | $1\times1\times2$ | $(= 2)$ |
| $3+2$ | $2\times3$ | $(= 6)$ | $2+1+1+1$ | $3\times1\times1\times1$ | $(= 3)$ |
| $2+3$ | $3\times2$ | $(= 6)$ | $1+2+1+1$ | $1\times3\times1\times1$ | $(= 3)$ |
| $2+2+1$ | $3\times3\times1$ | $(= 9)$ | $1+1+2+1$ | $1\times1\times3\times1$ | $(= 3)$ |
| $2+1+2$ | $3\times1\times3$ | $(= 9)$ | $1+1+1+2$ | $1\times1\times1\times3$ | $(= 3)$ |
| $1+2+2$ | $1\times3\times3$ | $(= 9)$ | $1+1+1+1+1$ | $1\times1\times1\times1\times1$ | $(= 1)$ |

$$S_5 = 3\times2+2\times6+3\times9+3\times2+4\times3+1 = 64 = 8^2$$

So far we seem to be obtaining perfect squares. A good number of students at this point GUESSED that one will always get a perfect square. A few students noticed a pattern,

$$S_n : \qquad 1^2, 2^2, 3^3, 5^2, 8^2, \ldots$$

The sequence $\sqrt{S_n}$ coincides, at least for $n \le 5$, with the famous *Fibonacci sequence*

$$F_n : \qquad 1, 2, 3, 5, 8, 13, 21, \ldots$$

Given the first two terms of the Fibonacci sequence, $F_1 = 1$, $F_2 = 2$, all other terms are found adding up the previous two terms, namely,

$$F_n = F_{n-1} + F_{n-2}.$$

**Conjecture:** $S_n = F_n^2$ for all $n > 0$.

At this point one could try one more experiment to validate the conjecture. Check by hand that

$$S_6 = 169 = 13^2 = F_6^2.$$

**Proof of the conjecture:** We will proceed by mathematical induction in a fashion very similar to Solution 2 for Problem 4(b) in the first round. We have already checked the cases $n = 1, 2, 3, 4, 5, 6$. Assume now that $S_k = F_k^2$ for all $k \le n$. We will show that $S_{n+1} = F_{n+1}^2$.

- There are $2^{n-1}$ ways to write $(n+1)$ as a sum of positive integers so that the last summand is 1,

  $$n+1 = (n)+1 \to (\text{corresponding product}) \times 1,$$

  where $(n)$ denotes one of the possible $2^{n-1}$ ways of writting $n$ as a sum of positive integers where the order matters. These terms when added up will contribute $S_n \times 1 = S_n$ to $S_{n+1}$.

- There are $2^{n-2}$ ways to write $(n+1)$ as a sum of positive integers so that the last summand is 2,

  $$n+1 = (n-1)+2 \to (\text{corresponding product}) \times 3,$$

  where $(n-1)$ denotes one of the possible $2^{n-2}$ ways of writting $n-1$ as a sum of positive integers where the order matters. These terms when added up will contribute $S_{n-1} \times 3 = 3S_{n-1}$ to $S_{n+1}$.

- There are $2^{n-k}$ ways to write $(n+1)$ as a sum of positive integers so that the last summand is $3 \le k \le n$,

  $$n+1 = (n-k+1)+k \to (\text{corresponding product}) \times 2,$$

  where $(n-k+1)$ denotes one of the possible $2^{n-k}$ ways of writting $n-k+1$ as a sum of positive integers where the order matters. These terms when added up will contribute $S_{n-k+1} \times 2 = 2S_{n-k+1}$ to $S_{n+1}$.

- Finally there is always the number itself, the case $k = n+1 \geq 3$,

$$n+1 = (0) + (n+1) \to 2,$$

which contributes $2S_0$ to $S_{n+1}$, where $S_0 = 1 = F_0 = F_0^2$.

Lo and behold,

$$\begin{aligned} S_{n+1} &= S_n + 3S_{n-1} + 2(S_{n-2} + S_{n-3} + \cdots + S_1 + S_0) \\ &= S_n + S_{n-1} + 2(S_{n-1} + S_{n-2} + S_{n-3} + \cdots + S_1 + S_0). \end{aligned}$$

We can now use the inductive hypothesis, $S_k = F_k^2$ for $k \leq n$,

$$S_{n+1} = F_n^2 + F_{n-1}^2 + 2(F_{n-1}^2 + F_{n-2}^2 + \cdots + F_1^2 + F_0^2).$$

**Claim:** $F_{n-1}^2 + F_{n-2}^2 + \cdots + F_1^2 + F_0^2 = F_{n-1}F_n$.

Assuming the claim is true, then,

$$S_{n+1} = F_n^2 + F_{n-1}^2 + 2F_{n-1}F_n = (F_n + F_{n-1})^2 = F_{n+1}^2,$$

and the conjecture is proved.

**Proof 1 of the claim:** We can proceed by induction once more. We should first check that the claim holds for $n = 1$,

$$F_0^2 + F_1^2 = 1 + 1 = 2 = F_1F_2.$$

It never hurts to check the next case for comfort (but it is really not necessary),

$$F_0^2 + F_1^2 + F_2^2 = 1 + 1 + 4 = 6 = F_2F_3.$$

Assume now that $F_0^2 + F_1^2 + \cdots + F_{n-1}^2 = F_{n-1}F_n$, show that $F_0^2 + F_1^2 + \cdots + F_{n-1}^2 + F_n^2 = F_nF_{n+1}$. By inductive hypothesis,

$$(F_0^2 + F_1^2 + \cdots + F_{n-1}^2) + F_n^2 = F_{n-1}F_n + F_n^2 = F_n(F_{n-1} + F_n) = F_nF_{n+1}.$$

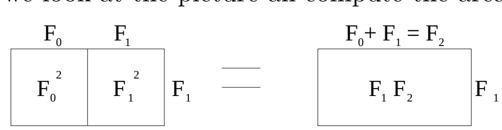
The claim has been proved.

**Proof 2 of the Claim (Prof. L.-S. Hahn):** This is a beautiful geometric argument. The idea is to interpret each summand on the right as the area of a square, and the term in the right as the area of a rectangle.

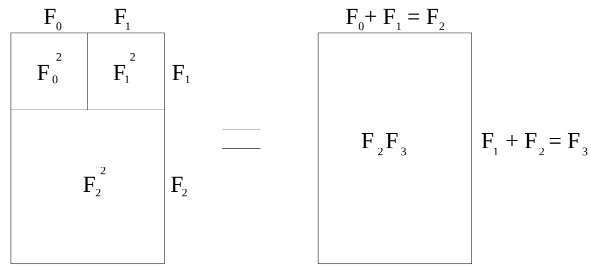
Consider the first identity we want to prove,

$$F_0^2 + F_1^2 = F_1F_2.$$

It is obviously true when we look at the picture an compute the areas in the two ways sketched.



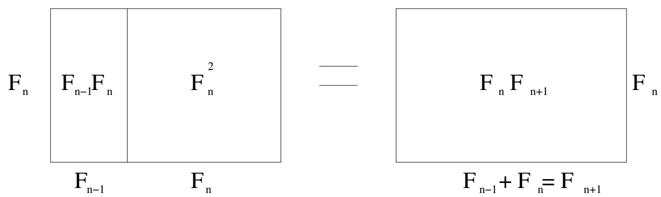
Add to the previous picture a square of sidelength $F_2$,




Then it is clear from the picture that

$$F_0^2 + F_1^2 + F_2^2 = F_2 F_3$$

Assume now that at step $n$ identity holds, hence if we draw a rectangle of sidelengths $F_n$ and $F_{n-1}$, its area coincides with $F_0^2 + \cdots + F_{n-1}^2$. Build a square on the side of length $F_n$, to obtain a new rectangle whose area is the area of the initial rectangle plus $F_n^2$, but at the same time its sidelenths are $F_n$ and $F_{n-1} + F_n = F_{n+1}$, hence its area is also equal to $F_n F_{n+1}$.




This problem is a creation of Prof. L.-S. Hahn. Only two students had a complete proof for this problem: **12th graders Jeff Dimiduk and Robert Cordwell (El Dorado HS).**

***

*Dear students: If you have any suggestions about the Contest, or if you have different solutions to any of this year's problems, please send them to:*


*Prof. Cristina Pereyra*
*Dept. of Mathematics and Statistics*
*University of New Mexico*
*Albuquerque, NM 87131*

*or e-mail them to:* *crisp@math.unm.edu*


*Remember that you can find information about past contests at:*
*http://www.math.unm.edu/math_contest/contest.html*


*I would like to express my gratitude for the invaluable help provided by*
*Prof. L.-S. Hahn.*
*His input made, as always, the exam better than it was originally*
*Finally thanks to all of the participants, their teachers and families,*
*you are an inspiration for us.*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