UNM - PNM STATEWIDE MATHEMATICS CONTEST XLII

November 7, 2009 First Round Three Hours

1. Let $f(n)$ be the sum of n and its digits. Find a number n such that $f(n) = 2009$.

   Answer: 1990

   If x, y, and z are digits, we can start with at most $200x$, but if we add $2+x$ to this, the answer will be even. Since the first digit must be 1 or 0, and the sum of the lower three digits is ≤ 27, we need to start with at least $198x$, but, again, if we add x, this gives an even number. So, we need to have a number of the form $199x$, and 1990 is the only one that works. (Given by NM Math Math Team)

2. What is the largest number of points that you can place on a circle of radius 1 so that the distance between any two points is strictly greater than $\sqrt{2}$?

   Answer: 3

   If we place four points on a circle, the least distance between adjacent points is maximized if the four points are the corners of a square. With radius 1, and each arc subtending $90°$, the distance between two adjacent points is exactly $\sqrt{2}$. Thus, three points will work, if each pair of points is more than $90°$ of arc apart. (Given by NM Math Team)

3. At a roadside stand, you can buy apples in bags of 6 or in bags of 11. What is the smallest number n such that you can buy exactly N apples for any $N \geq n$?

   Answer: 50

   We cannot buy exactly 49 apples as 6 does not divide any of the numbers 49, $49-11$, $49-2\cdot 11$, $49-3\cdot 11$ or $49-4\cdot 11$. We shall prove that if $N \geq 50$ we can buy exactly N apples. Observe that

$$0\cdot 11 = 0\cdot 6+0,\quad 1\cdot 11 = 1\cdot 6+5,\quad 2\cdot 11 = 3\cdot 6+4,$$
$$3\cdot 11 = 5\cdot 6+3,\quad 4\cdot 11 = 7\cdot 6+2,\quad 5\cdot 11 = 9\cdot 6+1.$$

   Let $N = 6q+r$ for some non-negative integers q and r with $0 \leq r \leq 5$. From the above observation it follows there is a $k \in \{0,1,2,3,4,5\}$, so that $11k = 6q_k + r$ for some non-negative integer q_k depending on k. Therefore

$$N = 11k + 6(q - q_k)$$

   and so it is enough to show that $q \geq q_k$. Suppose $q < q_k$. Then $6q < 6q_k$, hence

$$50 \leq N = 6q + r < 6q_k + r = 11k \leq 55,$$

   taking into account $0 \leq k \leq 5$. This is a contradiction, since the numbers $11k$ and N have the same remainder when divided by 6, so 6 divides $11k - N$ while this is a number between 0 and 5. Thus $q \geq q_k$ and we can buy exactly N apples.

4. David, Bill and George are three thieves. One of them committed a robbery. During the interrogation they made the following statements:
   - David: Bill is not the robber. I am not the robber.
   - Bill: David is innocent. George is the robber.
   - George: I am innocent. David is the robber.

   It was determined that one of them lied twice, one told the truth twice, and one lied once and told the truth once. Who is the robber?

   Answer: David

   If David is the robber, then David is lying once, Bill is lying twice and George is telling the truth twice. So David is the robber.

   Note that if Bill were the robber, then both David and Bill would be lying once only and if George were the robber, David and Bill would be telling the truth twice.

   Alternatively, we can consider all possible cases in order to find the robber. Case 1. David has two True statements, thus George is the robber. Then Bill has at least one True statement, so the other is false - a contradiction. Case 2. Bill has two True statements, thus George is the robber. Then David has two True statements - a contradiction. Case 3. George has two True statements and David is the robber. Then Bill has one True statement and one lie, while George has two lies. Case 3 is the only possible, so the robber is David.

5. Suppose $f(x) = ax + b$. If $f(f(f(x))) = 64x + 63$, what is $a + b$?

   Answer: 7

   $f(f(f(x))) = a(a(ax+b)+b)+b = a^3x+a^2b+ab+b$. So if $f(f(f(x))) = 64x+63$, then $a = 4$ and $(16 + 4 + 1)b = 63$, so $b = 3$. Thus $a + b = 7$.

6. Suppose a group of people have a code between themselves on how they can send messages to others in the group.
   - If A can send a message to B and B can send a message to C, then C can send a message to A.
   - For each pair of distinct people A and B in the group, either A can send a message to B or B can send a message to A but not both.

   What is the largest number of people in the group?

   Answer: 3

   Here are two possible explanations (one using relations and one using graph theory.)

   Solution 1: Let X be the group of people. Sending of messages among the group translates to a relation R on X (a subset of $X \times X$) such that if $(A, B) \in R$ and $(B, C) \in R$, then $(C, A) \in R$ and either $(A, B) \in R$ or $(B, A) \in R$ but not both. A relation with these properties forces X to have at most 3 elements, since any relation with 4 or more elements will have both (A, B) and (B, A) for some

A and B in X. Note that if $(A, B) \in R$ and $(B, C) \in R$ and $(B, D) \in R$, then $(C, A) \in R$ and $(D, A) \in R$. We only do not know the relationship between C and D. If $(C, D) \in R$, then $(D, B) \in R$ which cannot be the case since $(B, D) \in R$. If $(D, C) \in R$, then $(C, B) \in R$ which also cannot be the case.

Solution 2: Suppose that we have one node A pointing to two other nodes B and C. By assumption one of B or C points to the other. Without loss of generality, if B points to C, then, by another assumption, $A \to B \to C \Rightarrow C \to A$. But this gives a contradiction, since we would have A and C pointing towards each other. So, we know that no node can point to more than one other node.

Similarly, if we have a node that is pointed to by two other nodes, we will get a contradiction.

This means that we can have at most 3 nodes; otherwise we would end up with nodes that had edges pointing to more than one other node (each node connects to at least three other nodes, so there must be at least two edges pointing out, or two edges pointing in). (Given by NM Math Team)

Remark: Strictly speaking the statement of the problem does not say that every person communicates with someone. Thus, if we allow people who do not communicate with anyone, then we can have as many people as we want. Hence there will be no upper bound.

7. For which k does the system $x^2 - y^2 = 0$, $(x-k)^2 + y^2 = 2$ have exactly two real solutions?

Answer: $k = \pm 2$

Using the first equality, substitute x^2 for y^2 in the second equality. Thus, we are on the hunt to solve,

$$2x^2 - 2kx + k^2 - 2 = 0.$$

Using the quadratic formula,

$$x = \frac{2k \pm \sqrt{4k^2 - 8(k^2-2)}}{4} = \frac{k \pm \sqrt{4-k^2}}{2}.$$

To obtain a real solution, $-2 \leq k \leq 2$. There are exactly two solutions only if $4 - k^2 = 0$. Thus if $k = 2$ or $k = -2$, there are 2 solutions $\{(1,1),(1,-1)\}$ and $\{(-1,1),(-1,-1)\}$ respectively.

8. A library is open every day except Sunday. One day three girls, Adele, Bonnie and Clara visit the library together for the first time. Thereafter they visit the library many times. Adele makes her next visit two days after the previous visit unless the library is closed, in which she goes after three days. Bonnie makes her next visit three days after the previous visit unless the library is closed, in which she goes after four days. Clara makes her next visit four days after the previous visit unless the library is closed, in which she goes after five days. If their next meeting in the library was Friday. What day of the week was their first visit?

Answer: Wednesday

Let M stand for Monday, T for Tuesday, W for Wednesday, R for Thursday, F for Friday and S for Saturday. Notice that Adele will cycle through the days of the week as follows: $TRS\overline{MWF}$, where the bar means this pattern will repeat indefinitely. Bonnie will cycle through the week as follows: $WSTF\overline{MR}$. Clara will cycle through the week as follows: $R\overline{MFTSW}$. Since Bonnie can only go to the library on Friday during the first or second week, their next meeting in the library must occur in the first or the second week. It can't happen in the first week, since the only way that Clara will be in the library that first week is if she was in the library on Monday. But if they start on Monday, Bonnie will never be in the library on Friday. So it must be the second Friday. The only way that they can be in the library on the second Friday is if they started their library visits on Wednesday.

9. How many pairs of positive integers have greatest common divisor 3! and least common multiple 18!?

Answer: 2^6

We need to analyze the prime factorizations of 3! and 18!.

$$3! = 2 \cdot 3$$

$$18! = 2^{16} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$$

Notice that if a certain power of a prime appears in the prime factorization of each of two numbers then this power of a prime will appear in the prime factorization of the *gcd*. We are looking for all numbers a and b with $gcd(a, b) = 3 \cdot 2$, so $3 \cdot 2$ should appear in the prime factorization of a and b. Furthermore, at least one of the numbers a and b has 2, but no higher power of 2 in its prime factorization as otherwise the *gcd* will be divisible by a higher power of 2; the same observation is valid for the prime number 3. On the other hand, since the $lcm(a, b)$ is $18! = 2^{16} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$ each of the powers 2^{16}, 3^8, 5^3, 7^2, 11, 13 and 17 must appear in at least one of the prime factorization of the two numbers and no higher power appears in either of the prime factorization of the two numbers. Taking into account the restriction on the *gcd* it follows that each of powers 5^3, 7^2, 11, 13 and 17 must appear in the prime factorization of either a or b (but not both!). To sum up, what we have is that both a and b have $3 \cdot 2$ in their prime factorization, while each of the *seven* powers 2^{15}, 3^7, 5^3, 7^2, 11, 13 and 17 must appear in the prime factorization of either a or b. So the sought number is one-half the number of ways we can split the set of these last seven numbers into two disjoint subsets (notice that there are exactly two partitions giving the same pair) . This is the same as one-half the number of subsets of a set of seven elements since choosing a subset determines also its complement. The number of subsets of a set of seven element is 2^7, so the answer is 2^6.

10. (a) Consider a right triangle and a square inscribed in a given circle of radius R. For which of the two figures is the sum of squares of its sides the largest?
    (b) Among all triangles inscribed in a given circle of radius R, which triangle has the largest sum of squares of its sides?

Answer: (a) They are both the same. (b) An equilateral triangle.

(a) By the Pythagorean Theorem, for any right triangle, $a^2+b^2=c^2$ where a and b are the perpendicular legs and c is the hypotenuse. Any right triangle inscribed in a circle has the diameter as its hypotenuse. Hence $a^2+b^2=(2R)^2=4R^2$. Hence the sum of the squares of the sides for any right triangle is $8R^2$. The sides of a square inscribed in a circle are $\sqrt{2}R$. Hence, the sum of the squares of its sides will be $8R^2$ also.

(b) For any triangle with sides a, b and c, the law of cosines gives us that $a^2+b^2-2ab\cos\theta=c^2$, where θ is the angle between sides a and b. Consider a triangle inscribed in the circle with one of sides equal to c. The angle between the other two sides is the same, call it θ. Thus $a^2+b^2=2ab\cos\theta+c^2$ and among all such triangles (with a fixed side of length c) the sum $a^2+b^2+c^2$ is maximized when $ab\cos\theta$ is maximized , i.e., ab is maximized. This is the same as asking for a triangle inscribed in the circle with one of sides equal to c with maximal area (since the area is $\frac{1}{2}ab\sin\theta$). Expressing the area using the base and height formula, the maximal area is achieved by the triangle for which the vertex across the side c is as far as possible from the side c, i.e., an isosceles triangle inscribed in the circle with one of sides equal to c. It follows that given a non equilateral triangle inscribed in the circle we can find another inscribed triangle with bigger sum of squares of its sides. In particular, if there is a triangle with a maximal sum of squares of its sides it must be an equilateral triangle inscribed in the circle.

The fact that there is a triangle with sum of squares of its sides bigger than the sum of squares of the sides of any other triangle is a harder question to answer. Notice that not *every* problem of maximizing or minimizing a certain quantity has an answer in the sense that there might not be a maximum or minimum (the quantity is unbounded), but even in the case when the quantity has a bound it might be the case that there is no example for which this extreme value is achieved.

In our case, we can either show that after iterating the above symmetrization construction we converge to an equilateral triangle or proceed as follows. From the symmetrization argument it follows it is enough to show that the inscribed equilateral triangle has bigger sum of squares of its sides than the sum for an isosceles triangle inscribed in the circle. Let θ be the angle between the equal sides of an inscribed isosceles triangle. From the definitions of the sin and cos functions the sides of the isosceles triangle are $a=2R\cos(\theta/2)$, $a=2R\cos(\theta/2)$ and $c=2a\sin(\theta/2)=4R\cos(\theta/2)\sin(\theta/2)$., where R is the radius of the circle. The sum of squares of its sides is $S=16R^2\cos^2(\theta/2)\sin^2(\theta/2)+2\cdot 4R^2\cos^2(\theta/2)$. Using that $\cos^2(\theta/2)+\sin^2(\theta/2)=1$ and letting $t=\cos^2(\theta/2)$ we see that $S=8R^2t(3-2t)$, where from the definition of t we have $0<t<1$ as $0<\theta/2<\pi/2$. For an equilateral triangle we have $\theta = 60^o$ and thus $t = 3/4$, which is the point where the quadric $t(3-2t)$ achieves its maximum. So the maximum sum of squares is achieved for an inscribed equilateral triangle.

11. Suppose you are planning an expedition the goal of which is to take a letter to a remote destination that is nine days from your location. There is no food and water along the way so the team has to carry everything they need to survive. Each member of the team can carry enough food and water for six days. What is the minimum number of people that you have to put on the team in order for at least one of the them to reach the destination, while the remaining members either reach the destination or return safely home (meaning with enough supplies to survive)? Assume also that food and water cannot be left anywhere without someone staying at this location.

    Answer: 3 people (if food can be divided in half on a given day) otherwise 4.

    If the daily food supplies can be divided in half: Three people, a, b, and c, start out with full backpacks. After $1\frac{1}{2}$ days, they have each used up $1\frac{1}{2}$ days of supplies. c gives $1\frac{1}{2}$ days worth of supplies to each of a and b, filling their backpacks, and returns using the remaining $1\frac{1}{2}$ days of supplies in his own backpack. a and b go on another $1\frac{1}{2}$ days, when b gives a $1\frac{1}{2}$ days worth of supplies, filling a's backpack, and b returns using the remaining 3 days of supplies in his own backpack. (Given by NM Math Team.)

    If the food cannot be divided: Four people are enough, while three cannot achieve the goal. Travel four for one day, three for the next (while the first one is returning home), then two (while the second is returning), then finish with one with a full load of supplies (while the third is returning home). In this plan, the first, second and third person used only 5 person $\times$ days, while the last one uses all his supplies to reach the destination. Of course, we need to justify that three people cannot achieve the goal. We leave this to you.