UNM - PNM STATEWIDE MATHEMATICS CONTEST XLIV

February 4, 2012 Second Round Three Hours

1. How many 4 digit numbers with first digit 2 have exactly two identical digits (like 2011, or 2012)?

   Answer: 432. The repeated number is either a 2 or a digit in the set $\{0, 1, 3, 4, 5, 6, 7, 8, 9\}$. Let $A$ be the set of 4 digit numbers starting with 2 which have an additional 2 among its digits and no other repeated digits. We have 3 ways of choosing where to place the 2. The additional two digits can be assigned in $9 \times 8$ ways. Therefore, the number of elements of the set $A$ is $|A| = 3 \times 72 = 216$. Let $B$ be the set of 4 digit numbers starting with 2 which have a repeated digit among the remaining three digits. There are three ways of choosing where the two repeated digits among $\{0, 1, 3, 4, 5, 6, 7, 8, 9\}$ go. Since there are 9 possible digits to choose from, we have 27 ways of placing these digits. Once we have chosen this digit there are 8 remaining digits left to place in the other digit. Hence there are $27 \times 8 = 216$ ways of placing the digits in $B$. Since $A$ and $B$ are disjoint sets we can compute the number of elements which are either in $A$ or in $B$ as follows $|A \cup B| = |A| + |B| = 216 + 216 = 432$.

2. If 15 cows can eat all the grass on 3 acres of land in 6 days and 25 cows can eat all the grass on 4 acres in 4 days, how many cows can eat all the grass on 6 acres in 3 days?

   Answer: 45. The solution makes use of a number of assumptions which should have been stated in the formulation of the problem. Let $A$ be the initial *amount* of grass on one acre of land, assumed to be the same for every acre in the considered problem. Let $G$ be the rate at which the *grass grows* each day on each acre. We shall assume that this rate is constant throughout the problem. Let $E$ be the number of acres of grass *eaten* by each cow per day. Again, we shall assume that this rate is constant throughout the problem. Then let $X$ be the number of cows that will clear 6 acres in 3 days - this is the number we want to find. We can set up the following three equations expressing that the cows are eating the full amount of grass.

$$\begin{aligned} 15 \cdot E \cdot 6 &= 3A + 18G \\ 25 \cdot E \cdot 4 &= 4A + 16G \\ X \cdot E \cdot 3 &= 6A + 18G \end{aligned}$$

   For example, the first equation expresses the fact that the amount of grass eaten by 15 of the cows in 6 days is $15 \cdot E \cdot 6$ and this quantity of grass should be the same as the initial amount $3A$ on 3 acres plus $3 \cdot 6 \cdot G = 18G$-the grass that grew during the 6 days of feeding on these 3 acres. Notice that the first two equations involve three variables. However, if we divide the third equation by $E$ we see that $X$ can be computed from the last equation if we know the ratios $A/E$ and $G/E$. In view of the just made observation we simplify the above system to the following form.

$$\begin{aligned} A/E + 6 \cdot G/E &= 30 \\ A/E + 4 \cdot G/E &= 25 \\ X = 2 \cdot A/E + 6 \cdot G/E & \end{aligned}$$

   Subtracting the second equation from the first one, we obtain $2G/E = 5$, hence

$$A/E = 25 - 4 \cdot 5/2 = 15.$$

   Thus $X = 2 \cdot 15 + 6 \cdot 5/2 = 45$.

3. You are given a box full of 2012 flashlights which can be switched on or off by pressing the same button. The position of the button does not indicate if the light of the flashlight is on or off. Suppose,

you know that $k$ of these flashlights are turned on. While blindfolded you have to figure a way to split the flashlights in two groups which contain the same number of flashlights that are turned on. For this you are allowed to take out of the box any number of the given flashlights and press the on/off button as many times as you like. Remember, there is no way for you to know if the light of any particular flashlight is on or off.

Solution: Take any $k$ of the flashlights out and then press once the on/ off button of each one of these $k$ flashlights. Suppose $n$ of the chosen $k$ flashlights were on. So, among the randomly chosen $k$ flashlights $n$ are switched on while the remaining $k-n$ are off. After pressing the button (once!) of each of the chosen flashlights we have $k$ flashlights of which $n$ are switched off while the remaining $k-n$ are on. On the other hand, the box still has the remaining $k-n$ flashlights which were turned on, so we have the required division in two groups.

4. A stack of 2012 cards is labeled with the integers from 1 to 2012, with different integers on different cards. The cards in the stack are not in numerical order. The top card is removed from the stack and placed on the table, and the next card is moved to the bottom of the stack. The new top card is removed from the stack and placed on the table, to the right of the card already there, and the next card in the stack is moved to the bottom of the stack. The process - placing the top card to the right of the cards already on the table and moving the next card in the stack to the bottom of the stack - is repeated until all cards are on the table. It is found that, reading from left to right, the labels on the cards are now in ascending order: $1, 2, 3, \dots, 2011, 2012$. In the original stack of cards, how many cards were above the card labeled 2011?

Solution: We shall *work backwards* from when there are 2 cards left, since this is when the 2011 card is laid onto the table. When there are 2 cards left, the 2011 card is on the top of the deck. We shall depict this situation symbolically as the ordered pair

$$xo,$$

where $x$ stands for the card labeled 2011 and $o$ stands for any other of the cards (we are not interested in their labels). This is the card we shall be tracking in our reverse time process. In the forward movement the next move from the state $xo$ is to place the card $x = 2011$ on to the table. However, since we are working backwords our moves are: i) "move the bottom to the top" followed by ii) "add to the top a card from the table". Pictorially our deck will run through the following states in which the above two moves are displayed as: i) move the last symbol to the first position; ii) add a $o$ in front of the ordered sequence:

$$\begin{aligned} &\underline{xo} \mapsto ox \mapsto oox \mapsto \underline{xoo} \mapsto oxoo \mapsto ooxo \mapsto o\,ooxo \mapsto o\,ooox \mapsto oo\,ooox \\ &\mapsto \underline{xo\,oooo} \mapsto oxo\,oooo \mapsto oox\,oooo \mapsto ooox\,oooo \mapsto oooo\,xooo \mapsto o\,oooo\,xooo \\ &\mapsto o\,oooo\,oxooo \mapsto oo\,oooo\,oxoo \mapsto oo\,oooo\,ooxo \mapsto ooo\,oooo\,ooxo \mapsto ooo\,oooo\,ooox \\ &\mapsto \underline{xooo\,oooo\,oooo} \mapsto \dots \end{aligned}$$

Recall that in the above picture we are going backwards in time, so $\mapsto$ means going back; the first card in an ordered sequence $oo \dots oooo\,ooxo \dots oooo$ is on the top of the deck while the last card is at the bottom of the deck. Now we start counting. First notice that starting from the $xoo$ state every time the $x$ comes to first position the next move is to add an $o$ in front followed by adding an $o$ in front etc.. This means that if the $x$ is first followed by $k$ of the $o$'s, then the next time we see $x$ in the first position we would have added another $k$ of the $o$'s. Therefore our ordered sequences starting with $x$ will be of lengths 2- this is the "exceptional state", 3, $2 \cdot 3$, $2^2 \cdot 3$, $2^3 \cdot 3$, etc, i.e., the $n$-th time after the exceptional state we are going to see $x$ in first position there are $2^{n-1} \cdot 3$ cards in the deck. In other words the number of cards has doubled every time the $x$ appears on top since its last appearance. Since

$$512 = 2^9 < 1536/3 < 2^{10} = 1024$$

it follows that the last time we see $x$ in the first place there are $3 \cdot 512 = 1536$ cards in the deck. Continuing to go backwards, the $x$ will be moving down in both moves i) and ii). Since there are $2012 - 1536 = 476$ of the $o$'s we still need to add then the $x$ will move back to position $2 \cdot 476 = 952$ when we add the 46-th $o$ to the front of the sequence. Therefore there will be exactly 951 of the $o$'s in the end, meaning that there were 951 cards are above the one labeled 2011.

5. Add 13 to the month you were born multiplied by 10. Multiply the result by 10. Now add the day of the month you were born. If you tell me the number you obtained I will be able to tell the day and month you were born. Explain how I can do that. Notes: the month is expressed as a number between 1 to 12; the day of the month you were born is a number between 1 to 31.

   Solution: Let us write symbolically the operations that you are asked to do on the month $m$ and the day $d$ of your birth date.

$$m \mapsto 10m + 13 \mapsto 100m + 130 \mapsto 100m + 130 + d = N.$$

Working in reverse, given $N$, I can compute $N - 130 = 100m + d$. This shows that the last two digits of $N - 130$ will be your birth day, the first two your month.

6. Given a natural number $n$ let $S(n)$ be the sum the digits of $n$. For example $S(12304) = 1 + 2 + 3 + 0 + 4 = 10$. Show that for any polynomial function

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$$

with positive integer coefficients $a_0, a_1, \ldots, a_k$ we have that $S(p(n))$ will take on some value infinitely many times as $n$ runs through the set of natural numbers.

   Solution: Let $p(x) = a_0 + a_1 x + \cdots + a_k x^k$. Consider the values of $p$ on the powers of 10,

$$p(10^m) = a_0 + a_1 \cdot 10^m + a_2 \cdot 10^{2m} + \cdots + a_k \cdot 10^{km}.$$

We will give two solutions showing that $S(p(10^m))$ will take on some value infinitely many times. The first argument will show that this is indeed the case. The second argument will find a value, namely $S(a_0) + S(a_1) + \cdots + S(a_k)$, which will be taken on infinitely many times.

1st Solution: Start adding the numbers $a_0$, $a_1 \cdot 10^m$, $a_2 \cdot 10^{2m}$, etc. stopping to "think" before every addition. The goal of the "thinking" is to observe if the number you are to add next has zeros in all the digits of the number you have so far. If this is the case at every step then your sums are formed by using the given numbers $a_0$, $a_1$, etc. being placed one after the other to the left of the previous sum after placing some 0's first. In the end the digits of $p(10^m)$ are just the digits of $a_0, a_1, \ldots, a_k$ and zeros. So, $S(p(10^m)) = S(a_0) + S(a_1) + \cdots + S(a_k)$. However, if on some step the "next" number is not large enough (you don't move sufficiently to the left) then your sum will depend on the addition of some of the given numbers. In any case, your final sum is either $S(a_0) + S(a_1) + \cdots + S(a_k)$ or depends on the sums of the given numbers. Since we are given finitely many numbers overall we have finitely many possibilities of such "interfering" additions while the digits of $p(10^m)$ are being evaluated for infinitely many $m$'s. Therefore, some value must be taken on infinitely many times.

2nd Solution: Let us take a natural number $N$ so that $10^N$ is greater than any of the (finitely many) numbers $a_0$, $a_1$, $\ldots$, $a_k$,

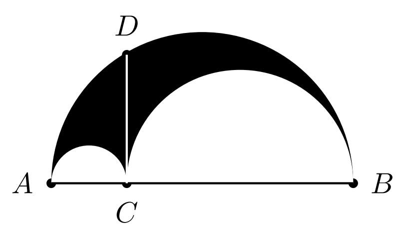
$$a_j < 10^N, \text{ for all } j.$$

For any $m \geq N$ and $1 \leq j \leq k-1$ then we have

$$\begin{aligned} a_0 + a_1 \cdot 10^m + a_2 \cdot 10^{2m} + \cdots + a_j \cdot 10^{jm} \\ \leq 10^N + 10^N \cdot 10^m + 10^N \cdot 10^{2m} + \cdots + 10^N \cdot 10^{jm} \\ = 10^N \left(1 + 10^m + 10^{2m} + \cdots + cdot 10^{jm}\right) \\ = 10^N \frac{10^{n(j+1)} - 1}{10^n - 1} = 10^{n(j+1)} - 1 \frac{10^N}{10^n - 1} < 10^{n(j+1)}, \end{aligned}$$

after using the formula for the sum of a geometric series $1 + q + q^2 + \cdots + q^j = (q^{j+1} - 1)/(q-1)$ and $q = 10^m$. The above calculation shows that $a_{j+1} \cdot 10^{m(j+1)}$ has zeros in all the digits of the number $a_0 + a_1 \cdot 10^m + a_2 \cdot 10^{2m} + \cdots + a_j \cdot 10^{jm}$. Since this is true for any $1 \leq j \leq k-1$ it follows $S(p(10^m)) = S(a_0) + S(a_1) + \cdots + S(a_k)$ for all $m \geq N$, which is what we wanted to prove.

7. Let $C$ be a point on the segment $AB$. Consider the region bounded by the three semicircles with diameters $AC$, $AB$ and $BC$ respectively. Show that the area of this region is equal to that of the disc with diameter the semi-chord $CD$ which is perpendicular to $AB$ in $C$.




Solution: This problem goes back to Archimedes. Let $a$, $b$ and $c$ be the lengths of $AC$, $CB$ and $CD$, respectively. In the first step we only use that the area of a disc of radius $r$ is $\pi r^2$. Therefore, the sought area is

$$A = \frac{\pi}{2}\left(\left(\frac{a+b}{2}\right)^2 - \left(\frac{a}{2}\right)^2 - \left(\frac{b}{2}\right)^2\right) = \frac{\pi}{4}ab.$$

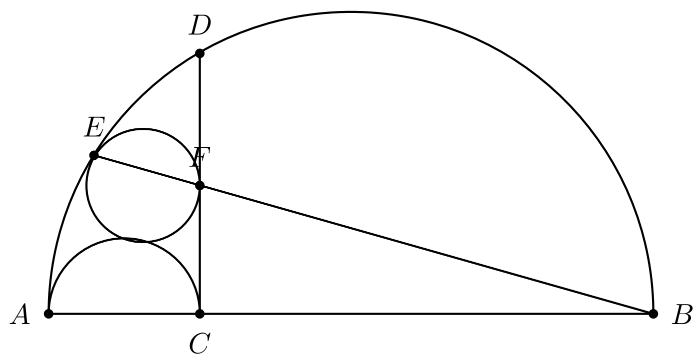
In the next step we use the properties of intersecting chords (or the formula for the height towards the hypothenuse of right triangle) from which we have $c^2 = ab$ using that $c$ is half of the length of the chord through $C$ and $D$. Thus the above formula for the area can rewritten as

$$A = \frac{\pi}{4}ab = \frac{\pi c^2}{4},$$

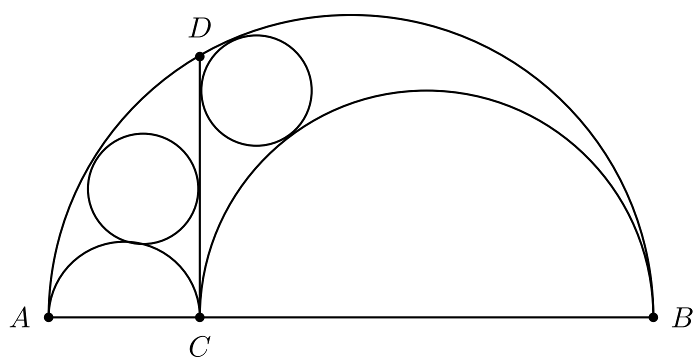
which is what we wanted to prove.

8. Let $C$ be a point on the segment $AB$. Consider the region bounded by the three semicircles with diameters $AB$, $AC$ and $CB$ respectively. The semi-chord $CD$ which is perpendicular to $AB$ at $C$ divides the semicircle in two smaller regions.

   (a) Inscribe a circle inside the smaller region containing $A$. Label the point where this circle intersects the arc $AD$ by $E$ and the point where the circle intersects the semi-chord by $F$. Show that $E$, $F$ and $B$ are collinear.

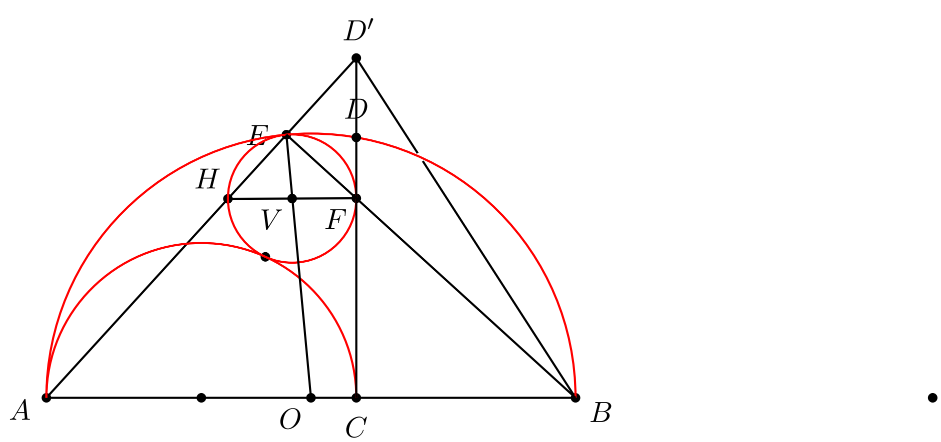



   (b) Show that the circles inscribed in the two regions bounded by the semicircles and the semi-chord are congruent (i.e., have same radii)



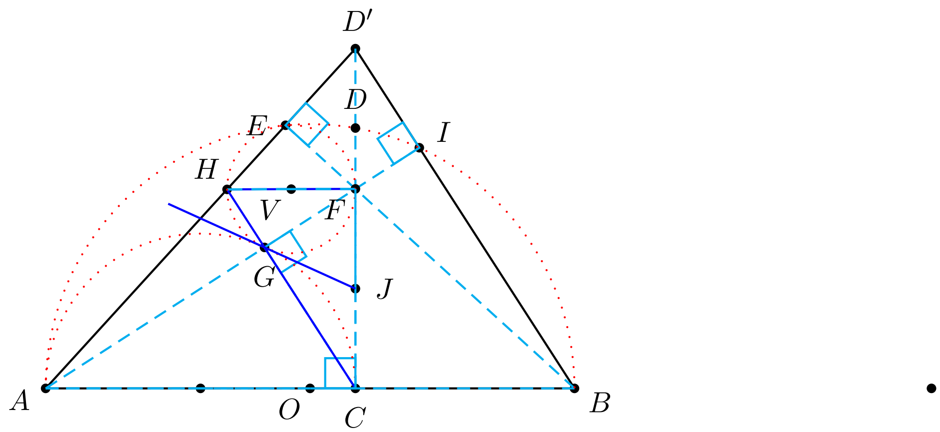

Solution: This problem goes back to Archimedes.

a) Let $V$ and $O$ be correspondingly the centers of the full circle in the "left" region and the circle with diameter $AB$ as shown on the picture below. Let $E$ be the common point of these two circles. Since the two circles are tangent at $E$ the radii $EO$ and $EV$ are both perpendicular to the common tangent line at $E$. Therefore, the lines through $EO$ and $EV$ are parallel, and since they have a common point $E$ the points $E$, $V$ and $O$ are co-linear.



Similarly, if $F$ is the common point of the circle (centered at $V$) and the tangent to it line $CD$ we have that $VF \perp CD$. Since we also have $AB \perp CD$ it follows that $VF \parallel AB$. Therefore the triangle $EVF$ is similar to the triangle $EOB$, $\triangle EVF \sim \triangle EOB$ taking into account that they are both isosceles and the angles between the equal sides are the same by the co-linearity of $E$, $V$ and $O$. This implies then that $\angle VEF = \angle OEB$. Therefore, $EF \parallel EB$. Since $EF$ and $EB$ have a common point it follows that the points $E$, $F$ and $B$ are co-linear.

b) Let $D'$ be the intersection of $AE$ and $CD$. Since $\angle AEF = \angle ACD' = 90^o$ it follows that $F$ is the orthocenter of triangle $ABD'$. Let $HF$ be the diameter through $F$ of the inscribed circle, hence $HF \perp CD'$. The point $H$ is on the segment $AE$ since $\angle HEF = 90^o = \angle AEB$ taking also into account part a).




Let $G$ be the intersection of the inscribed circle in the left region and the semicircle with diameter $AC$. Next we shall see that $G$ is also the intersection of the diagonals of the quadrilateral $ACFH$. For this it is enough to show that $\angle CGF = 90^o$ since then all the angles at $G$ are $90^o$. So, we consider the common tangent to the circles at the point $G$. Let $J$ be the intersection of this tangent and $CF$. By the property of the tangents we have

$$|JG| = |JF| = |JC|.$$

Therefore $G$ lies on the circle with diameter $CF$. This shows that $\angle CGF = 90^o$.

In the final step we shall determine $HF$ using similar triangles. In fact, we shall use $\triangle ACH \sim \triangle ABD'$ (since both $AF$ and $BD'$ are perpendicular to $HC$) and $\triangle D'HF \sim \triangle D'AC$. The similar triangle yield the identities

$$|AC|/|AB| = |AH|/|AD'| \quad \text{and} \quad |D'H|/|D'A| = |HF|/|AC|.$$

Therefore

$$|HF| = |AC| \cdot \frac{|D'H|}{|D'A|} = |AC| \cdot \frac{|D'A| - |AH|}{|D'A|} = |AC| \cdot (1 - |AC|/|AB|) = \frac{|AC| \cdot |CB|}{|AB|}.$$

In particular, the diameter of the "left" inscribed circle is determined by $|AC|, |CB|$ and $|AB|$, and depends in a symmetric way on $|AC|$ and $|CB|$. This shows that the same formula holds for the diameter of the circle inscribed in the "right" region. Therefore the two inscribed circles are congruent.

9. a) Find a number $R$ such that for any three points on or inside a square of side of length one at least two of the given points are at a distance at most $R$.

b) Determine the smallest number $R_0$, such that, given any three point on or inside a square of side of length one at least two of the given point are at distance at most $R_0$.

Solution: a) There are infinitely many choices. For example take $R = \sqrt{2}$. Since this is the length of the diagonal of the square any disc centered on the square of radius $R = \sqrt{2}$ contains the whole square. In particular given any two points on the square the distance between any two of them is at most $R$. Therefore, the condition that at least two of the given point are at distance at most $R$ is met. The point is that smaller $R$'s have this property as we shall see in part b).

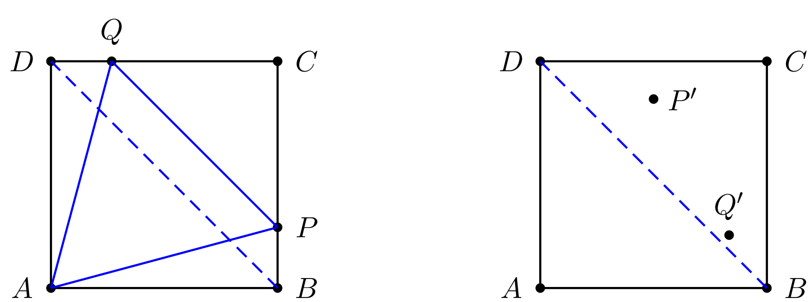
b) Consider, as shown on the picture below, the equilateral triangle $APQ$ of side of length $a$, where the lengths $|BP| = |QD|$. We can determine the value of $a$ by the Pythagorean theorem (for example). In fact, if we let $x = |BP| = |QD|$, then by the Pythagorean theorem applied to the right triangles $PCQ$ and $ADQ$ gives $a^2 = 2(1-x)^2 = 1^2 + x^2$, i.e.,

$$x^2 - 4x + 1 = 0.$$

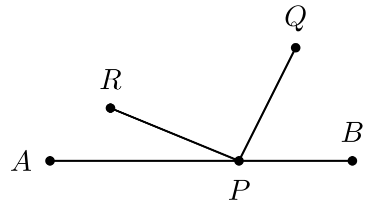
Solving for $x$, taking into account that $x > 0$ it follows $x = 2 - \sqrt{3}$. Hence

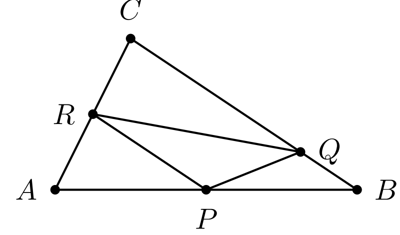
$$a = \sqrt{1 + (2-\sqrt{3})^2} = \sqrt{8 - 4\sqrt{3}} = 2\sqrt{2 - \sqrt{3}}.$$

We claim that $R_0 = a$. The fact that any number $0 < R < a$ does not have the needed property follows immediately by considering the three points $A$, $P$ and $Q$. On the other hand, $R_0 = a$ has the needed property, namely, given any three point on or inside the square of side of length one at least two of the given point are at distance at most $R_0$. We will show this fact as follows. Take any three points on or inside the square. If the distance between any two is less than or equal to $a$ then there is nothing to prove. Suppose all sides of the triangle formed by these three points are strictly greater than $a$. We will reach a contradiction in this case. Split the square in two parts by one if its diagonals. At least two of the picked points will lie in one of the halves. We can replace the third one by the opposite vertex of the square thereby increasing the distances between the "three" points. By renaming the vertices (or a rotation) of the square we can assume that the three points are $A$, $P'$ and $Q'$ as on the picture below, with $P'$ and $Q'$ sitting above or on the diagonal $BD$. Since the distances from $A$ to $P'$ and $Q'$ are greater than $a$ it must be that $P'$ and $Q'$ belong to the triangle $PQC$. However, this is a contradiction since then $|P'Q'| < a$.




10. a) Suppose $A$, $B$, $R$ and $Q$ be four points in the plane so that the points $R$ and $Q$ lie on one side of the line through the points $A$ and $B$. Determine the position of the point $P$ in the interior of the segment $AB$ so that the sum of the lengths of the segments $RP$ and $PQ$ is as small as possible.



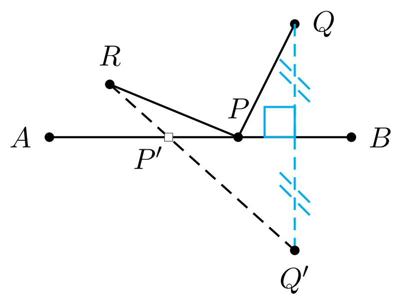




b) Let $ABC$ be a given acute triangle. Determine the triangle $PQR$ of smallest perimeter inscribed in the triangle$ABC$ so that the vertices $P$, $Q$ and $R$ lie, correspondingly, on the sides $AB$, $BC$ and $CA$ of the triangle $ABC$.

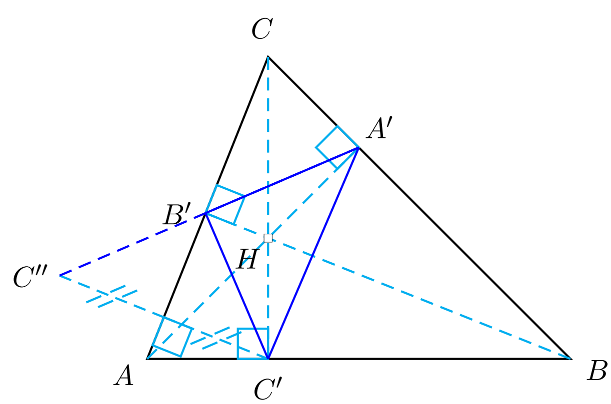
Solution: a) Let $Q'$ be the mirror image of $Q$ with respect to the line through $A$ and $B$. The required length is minimal when $P$ is the intersection $P'$ of $RQ'$ with $AB$. This is indeed the case since $|RQ'| = |RP'| + |P'Q|$, $|PQ| = |PQ'|$, hence by the triangle inequality we have

$$|RP'| + |P'Q| = |RQ'| \leq |RP| + |PQ'| = |RP| + |PQ|$$

with equality iff $P = P'$.




b) This part is known as Fagnano's Problem. A solution can be obtained by applying part a) using, correspondingly, each of the sides of the given triangle and the two vertices of the inscribed triangle that are not on this side. Having this in mind it follows that the perimeter of the triangle $PQR$ is as small as possible exactly when each of its vertices is on the segment connecting any of the other two vertices and the reflection of the remaining vertex with respect to the side containing the first vertex. This is indeed possible by taking the orthic triangle, i.e, the triangle $A'B'C'$ formed by the feet of the altitudes from $A$, $B$ and $C$ towards the opposite sides of the triangle $ABC$.




The claim on the needed property of the orthic triangle follows from the key fact that the altitudes $AA'$, $BB'$ and $CC'$ of the triangle $ABC$ are the (the angle) bisectors of the orthic triangle $A'B'C'$. Assuming the claim for the moment we note that it implies $\angle AB'C' = \angle CB'A'$, but since $\angle AB'C' = \angle AB'C''$ it follows

$$\angle AB'C' = \angle CB'A',$$

which shows that the points $C$", $B'$ and $A'$ are co-linear. We can argue similarly for the other sides or just relabel the vertices and apply the just made argument.

Next, we prove the claimed property of the orthic triangle. For this we use that if two right triangles have a common hypothenuse then the quadrilateral formed by the vertices of the triangles lie on the same circle (with diameter on the hypothenuse). This property implies the equality of some angles which we shall exploit repeatedly. For example, considering the quadrilateral $AC'HB'$ we have

$$\alpha = \angle B'AH = \angle B'C'H,$$

while the quadrilateral $BC'HA'$ yields

$$\beta = \angle A'BH = \angle A'C'H.$$

Now, the two right triangles $CA'A$ and $CB'B$ with a common angle at the vertex $C$ show that $\alpha = \beta$, i.e., $CC'$ is the angle bisect at $C'$ of the orthic triangle. Again, either by arguing similarly, or relabeling the vertices we obtain this property at the remaining vertices.