UNM - PNM STATEWIDE MATHEMATICS CONTEST XLVI

November 1-4, 2013    First Round    Three Hours

1. What are the last two digits of $7^{2013}$?

   Answer: The last two digits are 07.
   Solution: Note that $7^0 = 1$, $7^1 = 7$, $7^2 = 49$ and $7^3 = 343$ and $7^4 = 2401$. Hence, the last two digits cycle between $01, 07, 49$ and 43. So all powers which are congruent to 1 mod 4 will have last two digits 07. Since $2013 = 4(503) + 1$ then the last two digits of $7^{2013}$ will be 07.

2. Let $a \neq 0$ and $b$ be two given numbers. A quadratic equation $ax^2 + ax + b = 0$ has two solutions $x_1$ and $x_2$. What is the sum of the squares of the two solutions?

   Answer: $1 - \dfrac{2b}{a}$.
   Solution: Let $x_1$ and $x_2$ be the two solutions of the equation. Thus, we have $ax_1^2 + ax_1 + b = 0$ and $ax_2^2 + ax_2 + b = 0$ which show $a(x_1^2 + x_2^2) = -a(x_1 + x_2) - 2b$. Now we use Vieta's formulas for the sum of the roots, $x_1 + x_2 = -1$. In this case (and similarly in general case!), the Vieta's formulas follow from the expansion $ax^2 + ax + b = a(x - x_1)(x - x_2) = ax^2 - a(x_1 + x_2)x + ax_1x_2$ which shows $x_1 + x_2 = -1$ (and $x_1x_2 = b/a$). Thus

$$a(x_1^2 + x_2^2) = a - 2b, \quad \text{i.e.,} \quad x_1^2 + x_2^2 = 1 - 2b/a.$$

3. Simplify $\dfrac{2^4 + 2^4 + 2^4 + 2^4}{4^3 + 4^3 + 4^3}$.

   Answer: $\dfrac{1}{3}$.

$$\frac{2^4 + 2^4 + 2^4 + 2^4}{4^3 + 4^3 + 4^3} = \frac{4 \cdot 2^4}{3 \cdot 4^3} = \frac{2^2 \cdot 2^4}{3 \cdot (2^2)^3} = \frac{2^6}{3 \cdot 2^6} = \frac{1}{3}.$$

4. Suppose that on Halloween night three ghosts which are either spooks or poltergeists are having a conversation. Simple spooks always tell the truth but poltergeists always lie.
   Ghost 1: "Ghost 2 is a poltergeist."
   Ghost 2: "Ghost 1 and 3 are either both spooks or both poltergeists."
   What is Ghost 3?

   Answer: Poltergeist.
   Suppose Ghost 1 is a poltergeist. Then he is lying and Ghost 2 is a spook. Since spooks tell the truth, then Ghost 3 is a poltergeist like Ghost 1. Suppose Ghost 1 is a spook. Then he is telling the truth. So Ghost 2 is a poltergeist and a liar. Thus Ghost 3 must be a poltergeist. So in both cases, we see Ghost 3 is a poltergeist.

5. A palindrome is a number that reads the same forwards and backwards. How many 11 digit palindromes are there that include at most 2 of the same digit?

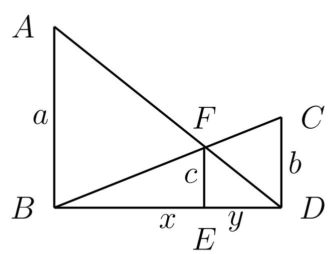
   Answer: $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$ or 136080.

   Solution: The middle (6th digit) can be arbitrary. Fixing the first five digits determines the last five digits. For the first digit we have 9 choices since it cannot be zero. For the second we have 9 choices again since it cannot be the same as the first, but can be zero. The third digit can be any among the unused 8 digits etc., until we reach the middle digit for which there are five choices.

6. Three trees in a perfectly flat forest are growing on the same line. The heights of the first and third trees are $a$ and $b$ respectively. What is the height of the middle tree if its top lies both on the line connecting the top of the first tree and the bottom of the third tree and the line connecting the top of the third tree with the bottom of the first tree?

   Answer: $\frac{ab}{a+b}$.

   Consider the following diagram where $AB = a$ and $CD = b$.




   We need to compute $c = EF$. Note that we have two sets of similar triangles: $\triangle ABD \sim \triangle FED$, and $\triangle CDB \sim \triangle FEB$. Thus we have

$$\frac{c}{a} = \frac{y}{x+y}, \quad \text{and} \quad \frac{c}{b} = \frac{x}{x+y}.$$

   Adding the two equations we obtain $c/a + c/b = 1$, which gives $c = ab/(a+b)$.

7. Find the largest sum of all integer numbers $x$ and $y$ such that

$$x^2y - xy^2 - xy - 2x + 2y + 1 = 0.$$

   Answer: 4.

   Solution: $x^2y - xy^2 - xy - 2x + 2y + 2 = (x-y-1)(xy-2)$, hence the integer solutions of $x^2y - xy^2 - xy - 2x + 2y + 2 = 0$ are given by the solutions of the systems

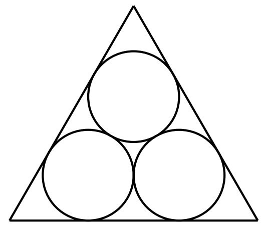
$$(i)\ x - y - 1 = 1, xy - 2 = 1 \ \text{ and } \ (ii) x - y - 1 = -1, xy - 2 = -1.$$

   For the system (i), the first equation gives $x = y + 2$ which we substitute in the second equation to find $(y+2)y = 3$, i.e., $0 = y^2 + 2y - 3 = (y+3)(y-1)$ which gives $y = 1$ and $x = 3$, or $y = -3$ and $x = -1$.

   For the system (ii), the first equation gives $x = y$ which we substitute in the second equation to find $y^2 = 1$. Thus, $x = y = 1$.

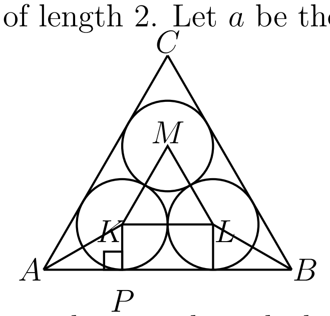
   The largest sum is 4 when $x = 3$ and $y = 1$.

8. Suppose three circles of radius one are packed inside an equilateral triangle so that every two circles are tangent to each other and the sides of the triangle are tangent to the circles; see figure below. What is the area of the triangle?



Answer: $\frac{\sqrt{3}}{4}(2+2\sqrt{3})^2 = \sqrt{3}(4+2\sqrt{3}) = 6+4\sqrt{3}$.

Solution: The smaller triangle $\Delta KLM$ with vertices at the centers of the packed circles is an equilateral triangle of side of length 2. Let $a$ be the side of the $\Delta ABC$.




By considering $\Delta APK$, which is a right triangle with the angle at $A$ equal to $30^o$ and side opposite $A$ equal to one, we see that $|AP| = \sqrt{3}$. This implies that $a = 2+2\sqrt{3}$ hence the area of $\Delta ABC$ is $A = \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}(2+2\sqrt{3})^2 = \sqrt{3}(4+2\sqrt{3}) = 6+4\sqrt{3}$

9. Simplify as much as possible the expression

$$\left(\frac{x(x^3+2y^3)}{x^3-y^3}\right)^3 + \left(\frac{y(y^3+2x^3)}{y^3-x^3}\right)^3.$$

Answer: $x^3+y^3$

Solution: Letting $x^3+2y^3 = u$ and $2x^3+y^3 = v$ we have $x^3-y^3 = v-u$, $x^3 = (2v-u)/3$ and $y^3 = (2u-v)/3$. Thus, we have

$$\left(\frac{x(x^3+2y^3)}{x^3-y^3}\right)^3 + \left(\frac{y(y^3+2x^3)}{y^3-x^3}\right)^3 = \frac{x^3(x^3+2y^3)^3}{(x^3-y^3)^3} - \frac{y^3(2x^3+y^3)^3}{(x^3-y^3)^3}$$

$$= -\frac{1}{3(u-v)^3}\left[(2v-u)u^3-(2u-v)v^3\right] = -\frac{1}{3(u-v)^3}\left[2vu^3-u^4-2uv^3+v^4\right]$$

$$= -\frac{1}{3(u-v)^3}\left[2uv(u^2-v^2)-(u^4-v^4)\right] = -\frac{1}{3(u-v)^3}\left[2uv(u^2-v^2)-(u^2-v^2)(u^2+v^2)\right]$$

$$= -\frac{1}{3(u-v)^3}(u^2-v^2)(2uv-u^2-v^2) = \frac{1}{3(u-v)^3}(u-v)(u+v)(u-v)^2$$

$$= \frac{1}{3}(u+v) = x^3+y^3.$$

10. The currency in Pairica uses bills of denominations 2, 4, 6,.., 100. In other words, there is a bill for every even integer number between 1 and 100. An urn contains 31 arbitrary bills

of different denominations. You are to name a number and if the urn contains two bills whose sum is the number you named you can take these two bills. What is the largest amount you can take with absolute certainty?

Answer: 122

Solution: There are 50 even numbers between 1 and 100. Take the smallest ten (i.e., 2,4,...,20) apart and consider the remaining 40. We can pair them so that the sum of the numbers of each pair is 122 - order them in increasing order and notice that the sum of the first (=22) and the last (=100), the 2nd (=24) and the 19th (=98), etc. is always 122. Thus we have 20 pairs each with a sum 122. If we pick 31 bills there will be at least one pair of which we must pick both numbers and it could be exactly one if we happen to pick the 10 numbers we set apart, one of each of the ten pairs and finally one more number of the remaining (which has to be a number from a pair). Thus the smallest number we can name and be guaranteed to get is 122.