UNM - PNM STATEWIDE MATHEMATICS CONTEST XLVII

February 7, 2015   Second Round   Three Hours

1. In the sequence
$$1, 2, 2, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8, \cdots$$
   what number occupies position 2015?

2. Show that if S is a set of finitely many non-collinear points in the plane (i.e., not all of the points are on the same line), then there is a line which contains exactly two of the points of S. Is the claim true if S has infinitely many points? Hint: Use an extremal configuration.

3. Show that the bisect of an angle in a triangle divides the opposite side in segments whose lengths have the same ratio as the ratio of the adjacent sides,
$$AN/NB = CA/CB$$
   in the picture below. NOTE: The same is true for the bisector of an exterior angle of a triangle, i.e., it divides the opposite side externally into segments that are proportional to the adjacent sides. You do not have to write a proof of this fact.
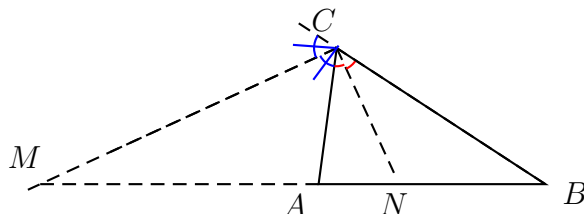
4. There are 12 coins in a parking meter and we know that one of them is counterfeit. The counterfeit coin is either heavier or lighter than the others. How can we find the fake coin and also if it is heavier or lighter in three weighings using a balance scale? Hint:4=3+1.

5. Let A and B be two points in the plane. Describe the set S of all points in the plane such that for any point P in S we have $|PA| = 3|PB|$.

6. A faulty calculator displayed $\diamond 38 \diamond 1625$ as an output of a calculation. We know that two of the digits of this number are missing and these are replaced with the symbol $\diamond$. Furthermore, we know that 9 and 11 divide the computed output. What are the missing digits and the complete output of our calculation?

7. Let A be the average of the three numbers $\sin 2\alpha$, $\sin 2\beta$ and $\sin 2\gamma$ where $\alpha + \beta + \gamma = \pi$. Express the product $P = \sin\alpha \sin\beta \sin\gamma$ in terms of A.

8. Suppose we draw circles of radius r with centers at every point in the plane with integer coordinates. What is the smallest r such that every line with slope $2/7$ has a point in common with at least one of these circles?

9. What is the probability of picking at random three points on a circle of radius one so that all three lie in a semicircle?

10. Solve $\underbrace{\overline{aa...a}}_{2k} - \underbrace{\overline{bb...b}}_{k} = \left(\underbrace{\overline{cc...c}}_{k}\right)^2$, where $\underbrace{\overline{cc...c}}_{k}$ denotes a number with k digits each one equal to c.