UNM - PNM STATEWIDE MATHEMATICS CONTEST XLVIII

February 6, 2016   First Round   Three Hours

1. Suppose there are 9 lights arranged on tic tac toe board so that one is in each square. Suppose further that there are six light switches one for each row and column. Flipping any of these switches turns on all lights that are off and turns off all lights that are in the column/row controlled by this switch. If there is exactly one light on, can you turn all the lights on using the given switches? As in all problems you need to explain your answer.

2. A student is offered two different after school jobs. One pays \$10 an hour, the other pays \$100 for the first hour, but the hourly rate decreases by half for each additional hour worked. What are the maximum number of hours the student can work at the second job, so that his total earnings in the second job are more than the total earnings in the first? Note: You are asked to find the maximum full hours of work (i.e. integer number) that maximize the earnings.

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3. A school purchased 4 peach trees, 4 apricot trees and 6 cherry trees that they want to plant in a row on the school grounds. If the trees are planted in random order, what is the probability that no two cherry trees are planted next to each other?

4. A farmer's house is in the shape of a convex pentagon with perimeter P and area A. The yard around the house includes all points that are at a distance at most $20m$ from the house. Find the area of the farmer's lot (yard plus house).

5. Show that if 19 points are chosen on a square of side of length 1 then there is a triangle with vertices among these points whose area is at most $\frac{1}{18}$.

6. For a positive integer k let $\sigma(k)$ be the sum of the digits of k. For example, $\sigma(1234) = 1+2+3+4 = 10$ while $\sigma(4) = 4$. Let $a_1 = 2016^{2016}$ and define $a_{n+1} = \sigma(a_n)$, $n = 1, 2, 3, \ldots$. Find a_5.

7. For a positive integer n let $S(n)$ denote the function which assigns the sum of all divisors of n. Show that if m and n are relatively prime positive integers then $S(mn) = S(m)\,S(n)$. For example, $S(6) = 1+2+3+6 = 12$, $S(2) = 1+2 = 3$ and $S(3) = 1+3 = 4$, so $S(6) = S(2)\,S(3)$, noting that 2 and 3 are relatively prime integers (they have no common divisor).

8. Find all non-negative integer solutions of the equation $n(n+1) = 9(m-1)(m+1)$.

9. Suppose every point in the plane is colored by one of two given colors, say red or blue. Given a triangle Δ, show that there is a triangle in the colored plane whose vertices are of the same color and is similar to the given triangle Δ.

10. Let P be a point on the triangle $\triangle ABC$ (inside or on the boundary). Let r_a, r_b and r_c be the distance from P to the sides BC, CA and AB, respectively.
    a) Show that
    $$r_a \cdot a + r_b \cdot b \leq |PC| \cdot c \quad \text{and also} \quad r_a \cdot b + r_b \cdot a \leq |PC| \cdot c,$$
    where $a = |BC|$, $b = |CA|$ and $c = |AB|$.
    b) Assuming the inequalities of part a), show that
    $$\frac{|PA| + |PB| + |PC|}{r_a + r_b + r_c} \geq 2.$$