UNM - PNM STATEWIDE MATHEMATICS CONTEST XLIX

November 4-7, 2016   First Round   Three Hours

1. Find all real solutions to the equality $x^4 - x^2 = 2 \cdot 2016$.

2. What is the last digit of 777^{777}?

3. A set of cards has an animal on one side and a number on the other. Seven cards are laid on a table and you can see a 1, 2 and a 4 and two cats, a mouse and a bird. If Erica says "Every card with a cat on one side has a 3 on the other side," how many cards will you have to flip to determine if Erica is telling the truth?

4. You would like to compute the square root of $S = 4$. Your friend suggests that you can use Hero's method for computing an approximation to $\sqrt{S}$. Hero's method produces a sequence, $x_0, x_1, x_2, \ldots$, of better and better approximations to $\sqrt{S}$ using the following algorithm. Given the current approximation, $x_i \approx \sqrt{S}$, the next approximation, x_{i+1}, is formed by taking the average of x_i and the quotient of S with x_i. If you start Hero's method with $x_0 = 1$ how many times do you have to repeat the process to get an answer with an error smaller than 10% of the correct answer?

5. Let $f(x) = \dfrac{x-1}{x+1}$. Determine $f(f(f(f(x))))$.

6. For your sister's 8th birthday, you decided to make her a cake in the shape of a regular octagon. Since you couldn't find a cake tin in this shape, you used an 8 inch diameter round cake tin and trimmed off the sides in such a way that you achieved the largest regular octagon possible. The area you cut off to form this octagonal cake is of the form $a\pi - b$ where a and b are real numbers. What is $\dfrac{b}{a}$?

7. There are five contestants on a game show, each one is given a distinct number between 1 and 5. The game they will play is as follows: Five separate closed boxes are placed in a room each has a labeled stone glued to the bottom. The boxes and the stones are each distinctly labeled with numbers between 1 and 5 and the stones have been glued randomly in the boxes. Each contestant is offered the chance to enter the room and look in three boxes. The contestants are kept separate during this process so as not to communicate the numbers on the stones in the boxes. The group wins if each of the five contestants sees a stone with his or her number on it. If the group is playing with the optimal strategy for winning the game, what is the group's probability of winning the game show prize money?

8. Find all integers $n > 1$ such that $4n + 9$ and $9n + 4$ are both perfect squares.

9. Let ABC be a triangle with sides $AC = 31$, $AB = 22$. Suppose the medians CC' and BB' are perpendicular. What is the length of BC?

10. In a factory there are three machines M_1, M_2, M_3 that are used in the production of two products P_1 and P_2. The production of one unit of P_1 occupies M_1 five minutes, M_2 three minutes and M_3 four minutes. The corresponding figures for P_2 are: M_1 one minute, M_2 four minutes and M_3 three minutes. The net profit per unit of P_1 is \$30 and for P_2 it is \$20. What is the maximal profit the company can make in an hour?