UNM - PNM STATEWIDE MATHEMATICS CONTEST XLIX

November 4-7, 2016   First Round   Three Hours

1. Find all real solutions to the equality $x^4 - x^2 = 2 \cdot 2016$.

   Answer: $x = \pm 8$
   Solution: Note that a factorization for $2016 = 4 \cdot 7 \cdot 8 \cdot 9$. Setting $x = \pm 8$, we see that $2016 = x/2 \cdot (x-1) \cdot x \cdot (x+1)$ or $x^4 - x^2 = 2 \cdot 2016$.
   Alternate solution given by Bill Cordwell:
   Let $y = x^2$ and note that y must be nonnegative if x is real. Completing the square gives $y^2 - y + \frac{1}{4} = (y - \frac{1}{2})^2 = 2 \cdot 2016 + \frac{1}{4} = \frac{8 \cdot 2016 + 1}{4} = \frac{16129}{4}$.
   This gives $y = x^2 = \frac{1}{2} \cdot \sqrt{16129} + \frac{1}{2} = \frac{1}{2} \cdot 127 + \frac{1}{2} = 64$, where we exclude the negative solution since y is nonnegative.
   Finally, solving for x gives $x = \pm 8$.

   To see ± 8 are the only solutions note that
$$x^4 - x^2 - 2 \cdot 2016 = (x^2 - 64)(x^2 + 63) = (x-8)(x+8)(x^2+63).$$
   Since $x^2 + 63$ has no real solutions, the only solutions are ± 8.

2. What is the last digit of 777^{777}?

   Answer: 7.
   Solution: We are looking for 777^{777} mod $10 = 7^{777}$ mod 10.
   Listing consecutive powers of 7, mod 10, gives
   $7^1 = 7$ mod 10
   $7^2 = 49 = 9$ mod 10
   7^3 mod $10 = 7 \cdot 9 = 63 = 3$ mod 10
   7^4 mod $10 = 7 \cdot 3 = 21 = 1$ mod 10, so $7^4 = 1$ mod 10.
   So, we need to find $7^{777 \ mod \ 4}$ mod $10 = 7^1$ mod $10 = 7$.

3. A set of cards has an animal on one side and a number on the other. Seven cards are laid on a table and you can see a 1, 2 and a 4 and two cats, a mouse and a bird. If Erica says“Every card with a cat on one side has a 3 on the other side,” how many cards will you have to flip to determine if Erica is telling the truth?

   Answer: 1-5 cards.
   Solution: One must flip the two cards with cats facing up and the cards with a 1, 2 and 4. If either of the cards with cats has a number other than 3, Erica is lying. If either of the numbered cards 1, 2, or 4 has a cat on the back, then Erica is lying. We do not need to flip the card with a mouse or a bird because Erica's statement doesn't pertain to cards with other animals besides cats.

4. You would like to compute the square root of $S = 4$. Your friend suggests that you can use Hero's method for computing an approximation to $\sqrt{S}$. Hero's method produces a sequence, $x_0, x_1, x_2, \ldots$, of better and better approximations to $\sqrt{S}$ using the following algorithm. Given the current approximation, $x_i \approx \sqrt{S}$, the next approximation, x_{i+1}, is formed by taking the

average of x_i and the quotient of S and x_i. If you start Hero's method with $x_0 = 1$ how many times do you have to repeat the process to get an answer with an error smaller than 10% of the correct answer?

Answer: 2. The iteration produces approximations $x_1 = 5/2$ with an error of $1/2$ or 25%, then $x_2 = 41/20$ with an error $1/20$ or 2.5%. It thus takes two repetitions.

Alternate solution: We have

$x_0 = 1$

$x_1 = \frac{1}{2}(1 + \frac{4}{1}) = \frac{5}{2} = 2.5$ (not close enough)

$x_2 = \frac{1}{2}(\frac{5}{2} + \frac{4}{(\frac{5}{2})} = \frac{1}{2}(\frac{5}{2} + \frac{8}{5}) = \frac{1}{2}(\frac{41}{10}) = 2.05$ (within 10% of $\sqrt{4} = 2$).

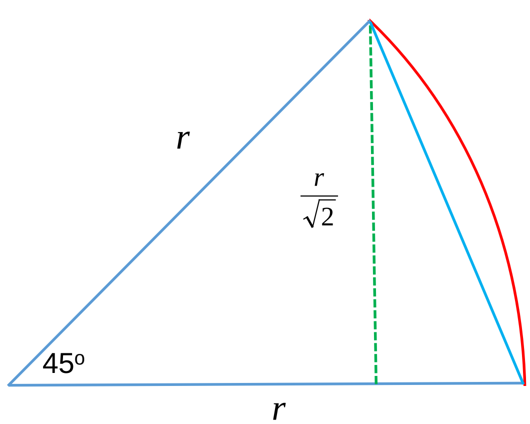
So, we have applied the procedure twice. Technically, we used the procedure initially and *repeated* it once, so (technically) we repeated the procedure once.

5. Let $f(x) = \dfrac{x-1}{x+1}$. Determine $f(f(f(f(x))))$.

   Answer: $f(f(f(f(x)))) = x$.
   Solution:
$$f(f(x)) = \frac{\frac{x-1}{x+1} - 1}{\frac{x-1}{x+1} + 1} = \frac{-2}{2x} = \frac{-1}{x}.$$
$$f(f(f(x))) = \frac{-1}{\frac{x-1}{x+1}} = \frac{-(x+1)}{x-1}.$$
$$f(f(f(f(x)))) = \frac{-(\frac{x-1}{x+1} + 1)}{\frac{x-1}{x+1} - 1} = \frac{-2x}{-2} = x.$$

6. For your sister's 8th birthday, you decided to make her a cake in the shape of a regular octagon. Since you couldn't find a cake tin in this shape, you used an 8 inch diameter round cake tin and trimmed off the sides in such a way that you achieved the largest regular octagon possible. The area you cut off to form this octagonal cake is of the form $a\pi - b$ where a and b are real numbers. What is $\dfrac{b}{a}$?

   Answer: $2\sqrt{2}$.



Solution: The area of the circle is $4^2\pi = 16\pi$. The area of the octagon is formed from 8 isosceles triangles each with 2 congruent sides of length 4 with interior angle 45 degrees (see picture above). So the area of each of these triangles will be $4\sin(45/2)4\cos(45/2) = 8\sin(45) = 4\sqrt{2}$. Since there are 8 such triangles, the area of the octagon is $32\sqrt{2}$. Hence, the area trimmed away will be $16\pi - 32\sqrt{2}$ meaning $a = 16$ and $b = 32\sqrt{2}$. Thus $\frac{b}{a} = 2\sqrt{2}$.

7. There are five contestants on a game show, each one is given a distinct number between 1 and 5. The game they will play is as follows: Five separate closed boxes are placed in a room each has a labeled stone glued to the bottom. The boxes and the stones are each distinctly labeled with numbers between 1 and 5 and the stones have been glued randomly in the boxes. Each contestant is offered the chance to enter the room and look in three boxes. The contestants are kept separate during this process so as not to communicate the numbers on the stones in the boxes. The group wins if each of the five contestants sees a stone with his or her number on it. If the group is playing with the optimal strategy for winning the game, what is the group's probability of winning the game show prize money?

Answer: 11/20

Solution: The optimal strategy is for each contestant to open the box with his or her number on it first then followed by the number they see on the stone in that first box. The third box they open will be the one which the stone in the second box directed them to. Note that the only stone placements which will fail the group will be those that shift the numbers in a cycle of 5 as illustrated below

1	2	3	4	5
2	3	4	5	1

or those that shift four of the numbers in a cycle of four like the following:

1	2	3	4	5
2	3	4	1	5

In the first scenario there are 24 such labelings since there are 4 ways of choosing the labeling the stone in box 1, 3 ways of choosing the labeling the stone in the box where the stone in 1 directed us, 2 ways of choosing the labeling the stone in the box where the stone in third box we looked at directed us and the last 2 stones must be placed in the boxes so that the stone and the box have different numbers.

In the second scenario there are 30 such labelings since since there are 5 ways of choosing the four boxes that will have their stones shifted. Then 3! ways of choosing the order of the shifting of the four boxes.

Since there are 120 ways of labeling the stones in the 5 boxes, then there are 66=120-24-30 ways of choosing labelings where the group would win the game. So the probability would be

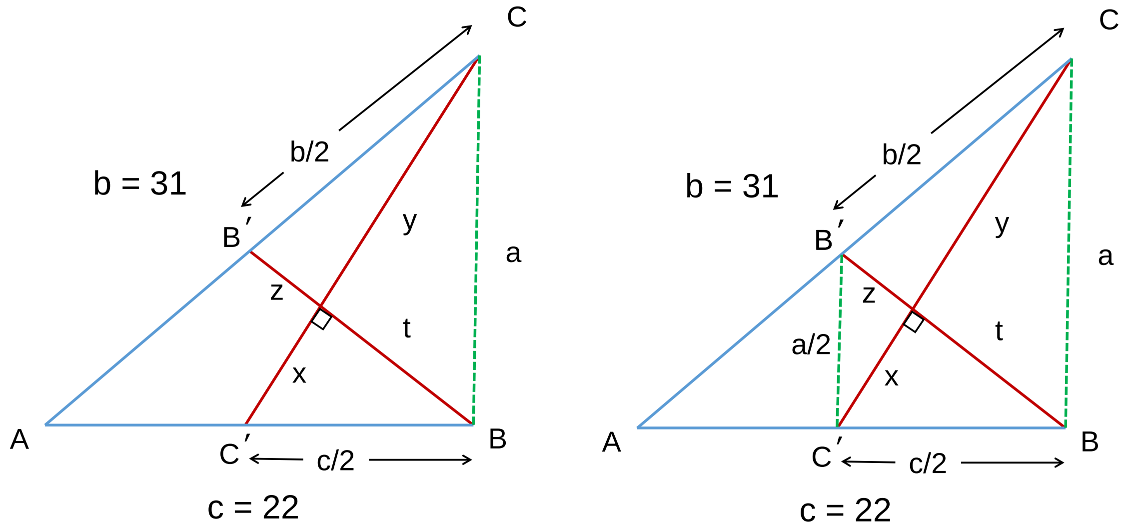
66/120 or 11/20.

8. Find all integers $n > 1$ such that $4n + 9$ and $9n + 4$ are both perfect squares.

   Answer: 28.
   Solution: If $4n + 9$ and $9n + 4$ are perfect squares then their product
$$(4n + 9)(9n + 4) = 36n^2 + (4^2 + 9^2)n + 36$$
is also a perfect square. Note that
$$(6n + 6)^2 \leq 36n^2 + (4^2 + 9^2)n + 36 < (6n + 9)^2$$
since $108 = 2 \cdot 6 \cdot 9 > 4^2 + 9^2 = 97$ but $96 = 2 \cdot 6 \cdot 8 < 97$. To find the n for which $(4n+9)(9n+4)$ is a perfect square, we set $(4n + 9)(9n + 4) = (6n + 6)^2$, $(4n + 9)(9n + 4) = (6n + 7)^2$ and $(4n + 9)(9n + 4) = (6n + 8)^2$ to obtain $n = 0$, $n = 1$ and $n = 28$. However, since $n > 1$, we see that n must be 28.

9. Let ABC be a triangle with sides $AC = 31$, $AB = 22$. Suppose the medians CC' and BB' are perpendicular. What is the length of BC?

   Answer: 17
   Solution: We illustrate the figure below
   Note that ABC is similar to $AB'C'$ with ratio 2 to 1. So $B'C' = 1/2BC$. Call the point of intersection of CC' and BB' D. Since CC' and BB' are perpendicular, then
$$(DC')^2 + DB^2 = 11^2$$
and
$$(DB')^2 + DC^2 = (31/2)^2.$$
Note that
$$DB^2 + DC^2 = BC^2$$
and
$$(DB')^2 + (DC')^2 = (B'C')^2 = 1/4BC^2.$$
Putting the above equalities together we obtain $5/4BC^2 = 121+961/4$ or $BC^2 = 1445/5 = 289$. Thus $BC = 17$.



10. In a factory there are three machines M_1, M_2, M_3 that are used in the production of two products P_1 and P_2. The production of one unit of P_1 occupies M_1 five minutes, M_2 three minutes and M_3 four minutes. The corresponding figures for P_2 are: M_1 one minute, M_2 four minutes and M_3 three minutes. The net profit per unit of P_1 is \$30 and for P_2 it is \$20. What is the maximal profit the company can make in an hour?

    Answer: 2250 dollar.

    Solution provided by Bill Cordwell:

    As the machines are independent of each other, we can look at them separately.

    M_1 can make one P_1 unit in five minutes compared to five P_2 units. Clearly, M_1 should be dedicated to making P_2 units, 60 P_2 units in an hour.

    M_2 can make one P_1 unit in three minutes compared to one P_2 unit in four minutes, and the P_1 units are more valuable, so M_2 should be dedicated to making P_1 units, 20 P_1 units in an hour.

    M_3 can make three P_1 units in 12 minutes, or it can make four P_2 units in 12 minutes. The profit is larger if M_3 makes P_1 units, 15 P_1 units in an hour.

    The maximum hourly profit is then $60 \cdot \$20 + (20 + 15) \cdot \$30 = \$2250$.

    Linear programming answer: If x_1 and x_2 units of P_1 and P_2 are produced per hour then the net profit is $f = 30x_1 + 20x_2$. The constraints are

$$\begin{aligned} 5x_1 + x_2 &\leq 60 \quad \text{for } M_1 \\ 3x_1 + 4x_2 &\leq 60 \quad \text{for } M_2, \\ 4x_1 + 3x_2 &\leq 60 \quad \text{for } M_3. \end{aligned}$$

    The profit is a plane so the maxima must be attained in one of the vertices in Figure 1. The maximal profit is attained in the vertex $x_1 = 120/11$, $x_2 = 60/11$.

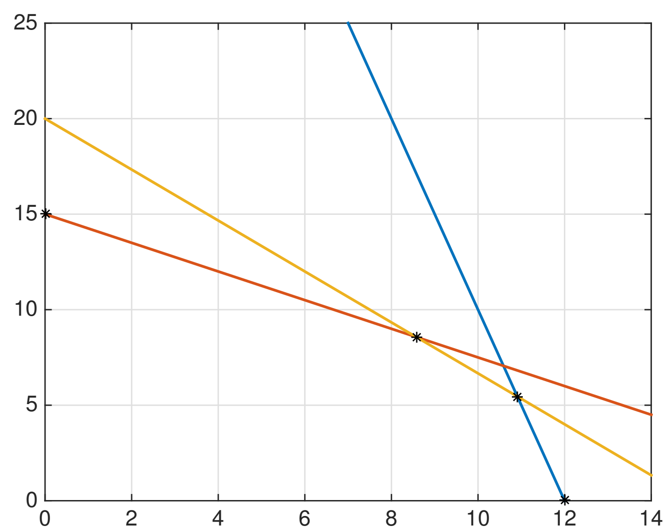


FIGURE 1. A linear program.