UNM - PNM STATEWIDE MATHEMATICS CONTEST L

February 3, 2018 Second Round Three Hours

1. Let $x \neq y$ be two real numbers. Let x, a_1, a_2, a_3, y and b_1, x, b_2, b_3, y, b_4 be two arithmetic sequences. Calculate $\frac{b_4-b_3}{a_2-a_1}$.

2. Determine all positive integers a such that $a < 100$ and $a^3 + 23$ is divisible by 24.

3. Let $a_1 < a_2 < a_3$ be three positive integers in the interval $[1, 14]$ satisfying $a_2 - a_1 \geq 3$ and $a_3 - a_2 \geq 3$. How many different choices of (a_1, a_2, a_3) exist?

4. Suppose $ABCD$ is a parallelogram with area $39\sqrt{95}$ square units and $\angle DAC$ is a right angle. If the lengths of all the sides of $ABCD$ are integers, what is the perimeter of $ABCD$?

5. Let x and y be two real numbers satisfying $x - 4\sqrt{y} = 2\sqrt{x-y}$. What are all the possible values of x?

6. A round robin chess tournament took place between 16 players. In such a tournament, each player plays each of the other players exactly once. A win results in a score of 1 for the player, a loss results in a score of -1 for the player and a tie results in a score of 0. If at least 75 percent of the games result in a tie, show that at least two of the players have the same score at the end of the tournament.

7. Let a, b be positive real numbers such that $\frac{1}{a} + \frac{1}{b} = 1$. Show that
$$(a+b)^{2018} - a^{2018} - b^{2018} \geq 2^{2\cdot 2018} - 2^{2019}.$$

8. Using red, blue and yellow colored toothpicks and marshmallows, how many ways are there to construct distinctly colored regular hexagons? (Note that two colored hexagons are the same if we can either rotate one of the hexagons and obtain the other or flip one of the hexagons about some line and obtain the other.)

9. Find the number of 4-tuples (a, b, c, d) with a, b, c and d positive integers, such that $x^2 - ax + b = 0$, $x^2 - bx + c = 0$, $x^2 - cx + d = 0$ and $x^2 - dx + a = 0$ have integer roots.

10. Let A, B, C and D be points in the Cartesian plane each a distance 1 from the origin $(0, 0)$. We define addition of points in the plane componentwise (If $P = (p_x, p_y)$ and $Q = (q_x, q_y)$, then $P + Q = (p_x + q_x, p_y + q_y)$). Show $A + B + C + D = (0, 0)$ if and only if A, B, C and D are the vertices of a rectangle.