UNM - PNM STATEWIDE MATHEMATICS CONTEST LI

February 2, 2019 Second Round Three Hours

1. A certain bacterium splits into two identical bacteria every second. A flask containing one such bacterium gets filled with the bacteria completely in 2019 seconds. Now, how long would it take the flask to get filled completely by the bacteria if it initially contained two bacteria?

2. A famous YouTuber, MathematicianJanet, is paid two thousand dollars every month as part of her sponsorship money. In addition, she makes nine dollars every time someone watches one of her videos on Youtube. If MathematicianJanet earned $16,107$ dollars in the last three months, how many of her videos were watched during those three months?

   (Note: YouTuber is a person who posts videos on Youtube.com. Other people can then watch these videos.)

3. Iris visited her grandparents during the summer vacations. Last summer, she saw six baskets with same number of eggs in her grandparents' house. Her grandma told her that if she took out fifty eggs from each basket, then the total number of the remaining eggs of the six baskets is equal to the number of eggs of the initial two baskets. Her grandma asked Iris if she knew how many eggs each basket has initially. Can you help Iris answer this question?

4. Let $\{a_n\}$ be a sequence of real numbers with the following properties:

   - $a_1 = 1$.
   - $a_{n+3} \leq a_n + 3$.
   - $a_{n+2} \geq a_n + 2$.

   Calculate a_{2019}.

5. Let x, y, z be positive real numbers. Suppose $x + y + z \geq xyz$, what is the minimum value of $u = \frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy}$?

6. Suppose p is a prime number and the number of positive factors of $p^2 + 71$ is no more than 10. What are the possible values of p?

7. Helen wants to approximate the perimeter of a circle but has forgotten the formula. Her friend Stephen suggests the following approximation to the perimeter of the circle: let the circle be inscribed in a n-sided regular polygon (recall: an inscribed circle is the largest possible circle that can be drawn on the inside of the polygon.) Then, approximate the perimeter of the circle by the perimeter of this n-sided regular polygon. What is the approximation to the perimeter of a circle with a diameter of 4 units if Helen uses Stephen's method with $n = 6$?

8. Four points are placed randomly on a circle. What is the probability that the quadrilateral formed by these points contains the center of the circle?

9. This question deals with equally-spaced points. Such points arise in many mathematical, engineering and computer science applications. We define the $n + 1$ equally-spaced points $x_i = ih$ where $h = 1/n$.

   (a) Prove that
   $$\text{if } 0 \leq j \leq n-1 \text{ then } (j+1)!(n-j)! \leq n!$$
   where $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$.

   (b) Establish the bound that for any $x \in [0, 1]$
   $$\prod_{i=0}^{n} |x - x_i| \leq \frac{1}{4} h^{n+1} n!,$$
   where $\prod_{i=0}^{n} z_i = z_1 \times z_2 \times \cdots \times z_n$. You may use the fact that for $x_j \leq x \leq x_{j+1}$ we have
   $$|x - x_j||x - x_{j+1}| \leq \frac{h^2}{4} \tag{1}$$

10. Here we have two circles C_1, C_2 with centers at O_1, O_2. C_1, C_2 intersect at two points A, B. P, E are two points in C_1; Q, F are two points in C_2. The line EF is a tangent line of C_1 and C_2. The line PQ is parallel to the line EF. The line PE intersects the line QF at the point R. Show that $\angle PBR = \angle QBR$.