UNM - PNM STATEWIDE MATHEMATICS CONTEST

November 2-5, 2018 First Round Three Hours

1. Consider the degree 9 polynomial $(4x-2)^9$. We can express this polynomial as $(4x-2)^9 = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + c_8x^8 + c_9x^9$. What is the sum $c_1+c_2+c_3+c_4+c_5+c_6+c_7+c_8+c_9$?

   Answer: 2^{10}

   Solution: For a polynomial $p(x) = \sum_{i=0}^{n} c_i x^i$, we have $p(0) = c_0$ and $p(1) = \sum_{i=0}^{n} c_i$. For our polynomial we have $p(1) = (4\times 1-2)^9 = 2^9$ and $p(0) = (0-2)^9 = -2^9$ and hence $\sum_{i=1}^{9} c_i = p(1) - p(0) = 2^9 + 2^9 = 2^{10}$.

2. How many positive integer factors of 9,800,000 are not perfect squares?

   Answer: 102

   Solution: We have $9,800,000 = 2^6 \times 5^5 \times 7^2$. We will count the number of factors that are perfect squares and then subtract from the total number of factors to obtain the number of factors that are not perfect squares. For perfect squares, we have 4 options for power of 2 $(0, 2, 4, 6)$, 3 options for power of 5 $(0, 2, 4)$ and 2 options for power of 7 $(0, 2)$ for a total of $4 \times 3 \times 2 = 24$ perfect squares. The total number of factors is $(6+1)(5+1)(2+1) = 126$. Hence the number of factors which are not perfect squares is $126 - 24 = 102$.

   Alternate Solution: In this solution we will count directly the number of factors that are not perfect squares. Each factor will be of the form $x = 2^a \times 7^b \times 5^c$, where we have 7 options for the power a of 2 (0,1,2,3,4,5,6), 3 options for the power b of 7 (0,1,2), and 6 options for the power c of 5 (0,1,2,3,4,5). The factor x will not be a perfect square provided that at least one of a, b, or c is odd.

   If $a = 1, 3$, or 5 (3 choices) regardless of the values of b and c, x is not a perfect square, and we have 3 choices for b (namely 0,1,2) and 6 choices for c (namely 0,1,2,3,4,5), that is a total of $3 \times 3 \times 6 = 54$ factors that are not perfect squares. If $a = 0, 2, 4$, or 6 (4 choices) and $b = 1$ then regardless of the value of c (6 choices) x is not a perfect square, there are $4 \times 6 = 24$ such factors. If $a = 0, 2, 4$, or 6 (4 choices) and $b = 0$ or 2 (2 choices) and $c = 1, 3, 5$ (3 choices) then x is not a perfect square, there are $4 \times 2 \times 3 = 24$ such numbers. This gives a total of $54 + 24 + 24 = 102$ factors of 9,800,000 that are not perfect squares.

Note that the two solutions indicate that for some counting problems its best to count the complement of the answer (as in the first solution) rather than directly counting the answer (as in the alternate solution).

3. List the number of two digit numbers $10 \leq x \leq 99$ satisfying the following property: x is divisible by the sum of its digits and x divided by the sum of its digits gives 7.

   Answer: Either 4 (the total number of such digits) or $21, 42, 63, 84$ (the list of such numbers).

   Solution: The question was worded ambiguously and so we accepted both the list of numbers or the total number of digits satisfying the requirements as correct answers. Since x is divisible by 7 and x is between 10 and 99, we know that x can be chosen from 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98. Among those numbers, the ones that satisfy the above property are: 21, 42, 63, 84.

   Alternate Solution: While listing all numbers divisible by 7 is easy for this problem, quite often this is not feasible so lets look at an algebraic way of solving this. Given x in between 10 and 99, $x = a + 10b$ where $0 \leq a \leq 9$ and $1 \leq b \leq 9$, a is the units digits, and b is the tenth digits. We need $x = 7(a + b)$, which leads to the equation $a + 10b = 7a + 7b$. Collecting a's on the left-hand-side and b's on the right-hand-side we conclude that $3b = 6a$, that is, $b = 2a$, and hence the tenth digit must be twice the units digit for the property to hold. The only values of a that will give permissible values of b are $a = 1, 2, 3, 4$ with corresponding values of $b = 2, 4, 6, 8$ (any other permissible value of a, namely $a = 0, 5, 6, 7, 8, 9$ yields unacceptable values of b given that $1 \leq b \leq 9$). These correspond to the four numbers of x we listed: $21, 42, 63, 84$.

4. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $a_1 = 0$ and
$$a_{n+1} = \frac{a_n - \sqrt{3}}{\sqrt{3}a_n + 1}$$
for $n = 1, 2, 3, \ldots$. Calculate $a_1 + a_2 + a_3 + \cdots + a_{2018}$.

   Answer: $-\sqrt{3}$.

   Solution: After direct calculations you observe that $a_2 = -\sqrt{3}$, $a_3 = \sqrt{3}$ and $a_4 = 0$ and from then on the pattern repeats every three terms. It is worth pointing out that you can now write a formula for the nth term depending solely on the remainder when dividing n by 3. In fact: $a_n = 0$ when $n = 3k+1$, $a_n = -\sqrt{3}$ when $n = 3k+2$, and $a_n = \sqrt{3}$ when $n = 3k$. In particular 2018 has remainder 2 when dividing by 3, since $2018 = 3 \times 672 + 2$, therefore $a_{2018} = -\sqrt{3}$, likewise since $2017 = 3 \times 672 + 1$, then $a_{2017} = 0$. Also note that the sum of any three consecutive terms is zero, in particular the sum of the first 2016 numbers is zero since is the sum 672 groups of three consecutive numbers:

$$(a_1 + a_2 + a_3) + (a_4 + a_5 + a_6) + ... + (a_{2014} + a_{2015} + a_{2016}) = 0,$$

therefore the sum of the first 2018 numbers equals the sum of the last two numbers, that is:

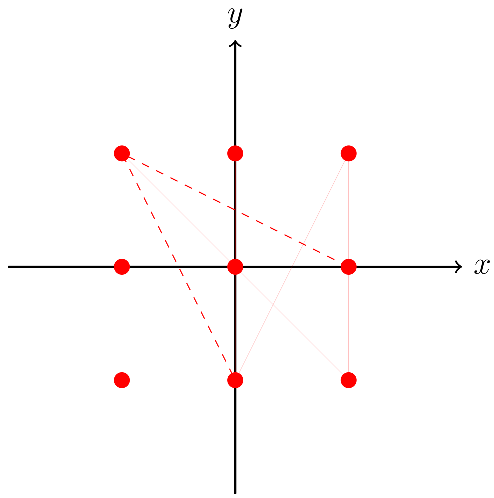
$$a_1 + a_2 + ... + a_{2018} = a_{2017} + a_{2018} = 0 - \sqrt{3} = -\sqrt{3}.$$

5. Let E be the subset of the plane consisting of points (x, y), where $x, y = -1, 0, 1$. For example the point $(-1, 0)$ is in E. Let us pick any three points from E. Then what is the probability that there exist two points among these three points whose distance is $\sqrt{5}$?

   Answer: $\frac{4}{7}$ or $\frac{64}{165}$.

   Note: There are two different answers to this problem. The first answer assumes that one can only pick three distinct points from E, the second answer assumes that one can pick same points from E.

   Solution 1: Assume that we can only pick distinct points from E. As shown in the picture below, E has 9 points. The ways to choose m points out of n, $m \leq n$ points are $\binom{n}{m} = \frac{n!}{m!(n-m)!}$, where $k! = 1 \times 2 \times 3 \times \cdots \times k$. So there are $\binom{9}{3}$ ways to pick 3 points out of 9 points. Notice that there are 8 pairs of points in E whose distance is $\sqrt{5}$. Two examples of such a pair are shown in the picture below. In order to have a pair of points whose distance is $\sqrt{5}$, , 2 pairs per each corner. We must pick a pair of points in these 8 pairs and then pick another point. So we have $8 \times 7 = 56$ choices. However, there are 8 cases which have been counted twice. See the picture below. So the probability is $\frac{56-8}{84} = \frac{4}{7}$.

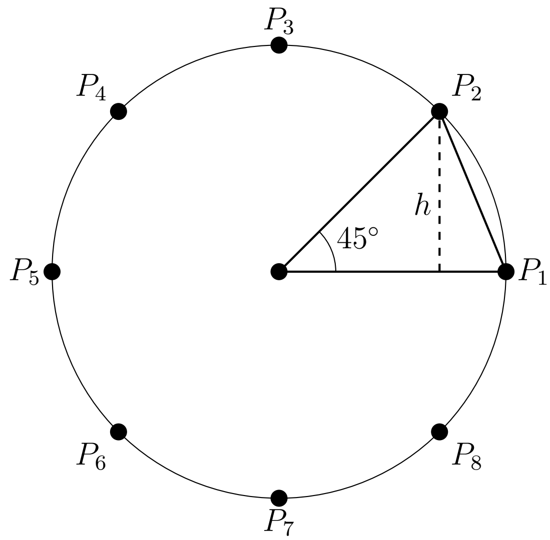


Solution 2: Assume that we can pick same points from E. As shown in the picture above, E has 9 points. Let us label those 9 points as 1, 2, 3, 4, 5, 6, 7, 8, 9. Each time when we pick three points $1 \le i \le j \le k \le 9$, we then let $i' = i,\ j' = j+1,\ k' = k+2$. Then we have $1 \le i' < j' < k' \le 11$. Thus to pick three points from nine points (those three points can be the same) is the same as to pick three distinct points from eleven points, which is $\binom{11}{3} = 165$. Notice that there are 8 pairs of points in E whose distance is $\sqrt{5}$. Two examples of such a pair are shown in the picture above. In order to have a pair of points whose distance is $\sqrt{5}$, we must pick a pair of points in these 8 pairs and then pick another point. So we have $8 \times 9 = 72$ choices. However, there are 8 cases which have been counted twice. See the picture above. So the probability is $\frac{72-8}{165} = \frac{64}{165}$.

6. King Hiero II of Syracuse wants to approximate the area of a circle but has forgotten the formula. His friend Archimedes suggests the following approximation to the area: take n equally spaced points $(P_1, P_2, \ldots, P_{n-1}, P_n)$ on the circumference of the circle resulting in a n-sided regular polygon. Then, approximate the area of the circle by the area of this n-sided regular polygon. What is the approximation to the area of a circle with a diameter of 4 units if King Hiero II uses Archimedes method with $n = 8$ points?

   Answer: $8\sqrt{2}$

   Solution: Divide the resulting octagon in 8 equivalent isosceles triangles with two sides of length 2 and angle between them of $\pi/4$ (or $45°$). The eight points and one such triangle are shown in the picture below. This gives that each triangle has base $b = 2$ and height $h = 2/\sqrt{2} = \sqrt{2}$ for an area of $\sqrt{2}$. Thus the approximate area of the circle is $8\sqrt{2}$.



7. There are two taps in a restroom which together fill a 100 gallon tank in 10 hours. However, Tap 1 takes 4 hours longer to fill a 50 gallon tank than it takes Tap 2 to fill a 30 gallon tank. What is the largest sized tank in gallons that can be filled by Tap 1 in 5 hours?

   Answer: 25 (gallons)

   Solution: Let Tap 1 pump t_1 gallons/hour and Tap 2 pump t_2 gallons/hour. Then, combined they pump $t_1 + t_2$ gallons in one hour or $10(t_1 + t_2)$ gallons in 10 hours. That is,

$$10(t_1 + t_2) = 100$$

   Let T be the time taken by Tap 1 to fill 50 gallons. Then $t_1 = 50/T$ and hence $t_2 = 30/(T - 4)$. Plugging these in above equation gives,

$$10\left(\frac{50}{T} + \frac{30}{T-4}\right) = 100$$

   Simplifying, we get $T^2 - 12T + 20 = 0$, or $(T - 2)(T - 10) = 0$. Hence T is either 2 or 10. However, given the statement of the problem, it takes Tap 1 at least 4 hours to fill 50 gallons, so $T = 10$. This gives $t_1 = 50/T = 5$. Hence in 5 hours, we can fill a tank of size $5 \times 5 = 25$ gallons.

8. Janet decides to donate money to a charity every year. But being a mathematician, she decides that the number of dollars she donates

to the charity would be equal to the total number of odd positive integers with distinct digits between 1 and the current year. (So, for example, 253 is one such number, but 799 is not since 9 is repeated). How much money did Janet donate to the charity this year (that is, in 2018)?

Answer: 592 (dollars)

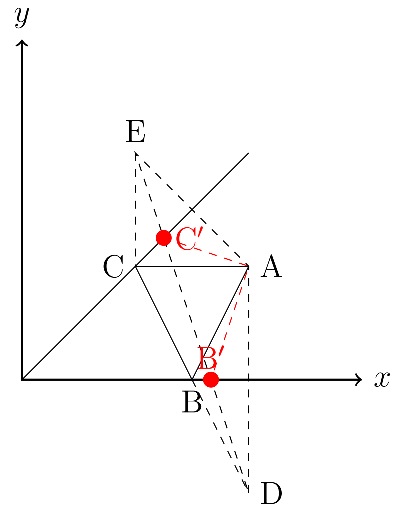
Solution: We look at cases.
(1) For numbers of the form 201X, there are 3 choices of X that work (X $\neq$ 9 or 1).
(2) For numbers of the form 200X, there are no choices.
(3) For numbers of the form 1XXX, we have 4 odd numbers possible for the last digit, times a remaining 8 for the second, times a remaining 7 for the third (we have used 1). This is 224.
(4) For numbers of the form XXX (no leading zeros), we have 5 for the last digit, 8 remaining for the first digit, and 8 for the middle one (it could be 0), or 320.
(5) XX gives 5 for the last digit, times 8 for the first (no zero) = 40.
(6) X gives 5 possibilities.

Hence, Janet donates 3 + 224 + 320 + 40 + 5 = 592 dollars.
(Special thanks to Bill Cordwell for this solution.)

9. Let A, B, C be three points on the plane. $A = (2, 1)$, the point B is on the x-axis, and the point C is on the line $x = y$. Suppose we can move B along the x-axis and C along the line $x = y$ freely. What is the minimum value of the perimeter of the triangle ABC?

Answer: $\sqrt{10}$.

Solution: Let us flip A along the x-axis to obtain a point $D = (2, -1)$ and flip A along the line $x = y$ to obtain a point $E = (1, 2)$. Notice that by symmetry the segments AB and BD have equal length and same holds for the segments AC and CE. Therefore the perimeter of triangle ABC is equal to $|EC| + |BD| + |CB|$. Note that this is the length of the polygonal "path" $ECBD$, joining E and D, but the shortest such path is the segment ED joining them. Let B' and C' be the intersection points of the segment ED and the lines $y = 0$ and $y = x$ and note that this configuration yields the smallest perimeter for the triangle $AB'C'$ because that perimeter is exactly $|ED|$.



10. Let x_1, x_2, x_3 be nonnegative real numbers such that $x_1+x_2+x_3 = 1$. Calculate the maximum value of

$$(x_1 + 3x_2 + 5x_3)(x_1 + \frac{x_2}{3} + \frac{x_3}{5}).$$

Answer: $\frac{9}{5}$.

Solution:

$$\begin{aligned}
&(x_1 + 3x_2 + 5x_3)(x_1 + \frac{x_2}{3} + \frac{x_3}{5}) \\
=&\frac{1}{5}(x_1 + 3x_2 + 5x_3)(5x_1 + \frac{5x_2}{3} + x_3) \\
\leq&\frac{1}{5}\frac{1}{4}\left((x_1 + 3x_2 + 5x_3) + (5x_1 + \frac{5x_2}{3} + x_3)\right)^2 \\
=&\frac{1}{20}\left(6x_1 + \frac{14}{3}x_2 + 6x_3\right)^2 \\
\leq&\frac{1}{20}(6x_1 + 6x_2 + 6x_3)^2 \\
=&\frac{9}{5}.
\end{aligned}$$

There are two inequalities that we used here: the first one is the geometric-arithmetic inequality and the second one is a majorization inequality. The geometric-arithmetic inequality (in our case) states that if one has two nonnegative real numbers a and b, then

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

Let us give a short proof here. Notice that $(a-b)^2 \geq 0$. Expand it and we have $a^2+b^2 \geq 2ab$ which implies $a^2+b^2+2ab \geq 4ab$. That is $(a+b)^2 \geq 4ab$. Take the square root at both sides and divide both sides by 2, we obtain the geometric-arithmetic inequality. Note that equality holds if and only if $a = b$. In our case the geometric-arithmetic inequality has been used with $a = x_1 + 3x_2 + 5x_3$ and $b = 5x_1 + 5x_2/3 + x_3$.

The second equality holds if and only if $x_2 = 0$. The first equality holds if and only if $x_1 + 3x_2 + 5x_3 = 5x_1 + \frac{5x_2}{3} + x_3$. Together with $x_1 + x_2 + x_3 = 1$, we obtain $x_1 = x_3 = \frac{1}{2}$ and $x_2 = 0$. So the maximum value is $\frac{9}{5}$.