UNM - PNM STATEWIDE MATHEMATICS CONTEST

February 2, 2019 Second Round Three Hours

1. A certain bacterium splits into two identical bacteria every second. A flask containing one such bacterium gets filled with the bacteria completely in 2019 seconds. Now, how long would it take the flask to get filled completely by the bacteria if it initially contained two bacteria?

   Answer: 2018 seconds.

   Solution: 2018 seconds. If we start with one bacterium, after one second we have two bacteria, which is the case we want. It took 2019 seconds when starting from one bacterium, which means starting from two bacteria must have taken 2018 seconds.

2. A famous YouTuber, MathematicianJanet, is paid two thousand dollars every month as part of her sponsorship money. In addition, she makes nine dollars every time someone watches one of her videos on Youtube. If MathematicianJanet earned $16,107$ dollars in the last three months, how many of her videos were watched during those three months?

   (Note: YouTuber is a person who posts videos on Youtube.com. Other people can then watch these videos.)

   Answer: 1123.

   Solution: She earned $16,107 - 6,000 = 10,107$ from videos watched. Total videos watched were $10,107/9 = 1123$.

3. Iris visited her grandparents during the summer vacation. Last summer, she saw six baskets with same number of eggs in her grandparents' house. Her grandma told her that if she took out fifty eggs from each basket, then the total number of the remaining eggs of the six baskets is equal to the number of eggs of the initial two baskets. Her grandma asked Iris if she knew how many eggs each basket has initially. Can you help Iris answer this question?

   Answer: 75.

   Solution: Iris' grandma took out $50 \times 6 = 300$ eggs which is equal to the number of eggs of the initial four baskets. So initially each basket has 75 eggs.

4. Let $\{a_n\}$ be a sequence of real numbers with the following properties:

   - $a_1 = 1$.
   - $a_{n+3} \leq a_n + 3$.
   - $a_{n+2} \geq a_n + 2$.

   Calculate a_{2019}.

   Answer: $a_{2019} = 2019$.

   Solution: Notice that
   $$a_{2019} \geq a_{2017} + 2 \geq a_{2015} + 4 \geq \cdots \geq a_1 + 2018 = 2019.$$
   Also we have
   $$a_n \leq a_{n+2} - 2 \leq a_{n-1} + 1.$$
   Then we have
   $$a_{2019} \leq a_{2018} + 1 \leq a_{2017} + 2 \leq \cdots \leq a_1 + 2018 = 2019.$$

5. Let x, y, z be positive real numbers. Suppose $x+y+z \geq xyz$, what is the minimum value of $u = \frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy}$?

   Answer: $u_{\min} = \sqrt{3}$.

   Solution: Since $x, y, z > 0$ and $x+y+z \geq xyz$, so

$$\frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz} \geq 1.$$

   Notice that by the geometric-arithmetic inequality, we have

$$\frac{x}{yz} + \frac{y}{xz} \geq \frac{2}{z}.$$

   Also

$$\frac{1}{x^2} + \frac{1}{y^2} \geq \frac{2}{xy}, \quad \frac{1}{x^2} + \frac{1}{z^2} \geq \frac{2}{xz}, \quad \frac{1}{y^2} + \frac{1}{z^2} \geq \frac{2}{yz}.$$

   Thus

$$\begin{aligned} u &= \frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} \\ &\geq \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \\ &= \sqrt{\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + 2\left(\frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz}\right)} \\ &\geq \sqrt{3\left(\frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz}\right)} \\ &\geq \sqrt{3}. \end{aligned}$$

   The above equalities holds if and only if $x = y = z$ and $\frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz} = 1$, i.e., $x = y = z = \sqrt{3}$. Thus $u_{\min} = \sqrt{3}$.

6. Suppose p is a prime number and the number of positive factors of $p^2 + 71$ is no more than 10. What are the possible values of p?

   Answer: $p = 2, 3$.

   Solution: When $p = 2$, $p^2 + 71 = 75 = 3 \times 5^2$, the total number of factors of 75 is $2 \times 3 = 6$.
   When $p = 3$, $p^2 + 71 = 80 = 2^4 \times 5$, the total number of factors of 80 is $5 \times 2 = 10$.
   When $p > 3$, $p^2 + 71 = (p-1)(p+1) + 72$. Since p is an odd number, $(p-1)(p+1)$ is divisible by 4. Since p is not divisible by 3, $(p-1)(p+1)$ must be divisible by 3. Thus $p^2 + 71$ is divisible by 24. Let $p^2 + 71 = 24 \times m$, where $m \geq 4$.

   - If m is divisible by a prime number greater than 3, then the total number of factors of $p^2 + 71$ is at least $4 \times 2 \times 2 = 16$.
   - If m is divisible by 3, then the total number of factors of $p^2 + 71$ is at least $4 \times 3 = 12$.
   - If $m = 2^k, k \geq 2$, then the total number of factors of $p^2 + 71$ is $(4 + k) \times 2 \geq 6 \times 2 = 12$.

   We thus conclude that $p = 2, 3$ are the only possible values.

7. Helen wants to approximate the perimeter of a circle but has forgotten the formula. Her friend Stephen suggests the following approximation to the perimeter of the circle: let the circle be inscribed in a n-sided regular polygon (recall: an inscribed circle is the largest possible circle that can be drawn on the inside of the polygon.) Then, approximate the perimeter of the circle by the perimeter of this n-sided regular polygon. What is the approximation to the perimeter of a circle with a diameter of 4 units if Helen uses Stephen's method with $n = 6$?

Answer: $24/\sqrt{3}$.

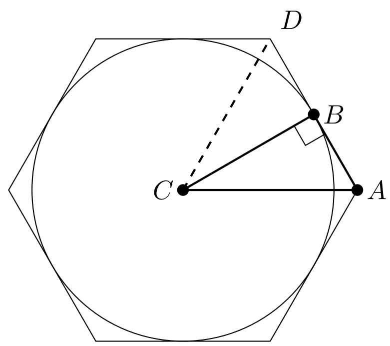
Solution:

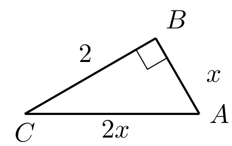
The regular hexagon can be divided into 12 congruent right-triangles. One such triangle ABC is indicated in the figure below. The perimeter of the hexagon is $12 \times x$ where x is the length of the line segment AB. We are given that length of line segment BC is 2.

Also note that triangle ACD is an equilateral triangle with length of each side equal to twice of length of line segment AB, that is equal to $2x$. Thus, length of line segment AC is $2x$. By the Pythagorean theorem on triangle CAB,

$$x^2 + 2^2 = (2x)^2$$

Solving this gives $x = 2/\sqrt{3}$. The perimeter of the hexagon is then $24/\sqrt{3}$.





8. Four points are placed randomly on a circle. What is the probability that the quadrilateral formed by these points contains the center of the circle?

Answer: 1/2.

Solution: Let us label the points as A,B,C and D. Define the clockwise semi-circle (CSC) of a point as the part of the circle starting from the point and ending at its antipode while traversing the circle in the clockwise direction. Then the probability of the quadrilateral not containing the center is

Probability of B, C, D being contained in the clockwise semi-circle of A

+

Probability of A, C, D being contained in the clockwise semi-circle of B

+

Probability of A, B, D being contained in the clockwise semi-circle of C

+

Probability of A, B, C being contained in the clockwise semi-circle of D

Each of these probabilities is the same. Let us compute one of them, say,

Probability of B, C, D being contained in the clockwise semi-circle of A
$= (\text{Prob. of B in CSC of A}) \times (\text{Prob. of C in CSC of A}) \times (\text{Prob. of D in CSC of A})$
$= (1/2) \times (1/2) \times (1/2) = 1/8$

Therefore, the probability of the of the quadrilateral not containing the center is $4 \times (1/8) = 1/2$. And hence the probability of the quadrilateral containing the center is $1 - 1/2 = 1/2$.

9. This question deals with equally-spaced points. Such points arise in many mathematical, engineering and computer science applications. We define the $n + 1$ equally-spaced points $x_i = ih$ where $h = 1/n$.

(a) Prove that

$$\text{if } 0 \le j \le n-1 \text{ then } (j+1)!(n-j)! \le n!$$

where $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$.

(b) Establish the bound that for any $x \in [0,1]$

$$\prod_{i=0}^{n} |x - x_i| \le \frac{1}{4} h^{n+1} n!,$$

where $\prod_{i=0}^{n} z_i = z_1 \times z_2 \times \cdots \times z_n$. You may use the fact that for $x_j \le x \le x_{j+1}$ we have

$$|x - x_j||x - x_{j+1}| \le \frac{h^2}{4} \tag{1}$$

Solution:

(a) First note that this equality is obviously true for $j = 0$. For $0 < j \le n-1$, we will show that,

$$\frac{(j+1)!(n-j)!}{n!} \le 1$$

The left hand side simplifies to,

$$\begin{aligned}
\frac{(j+1)!(n-j)!}{n!} &= \frac{(j+1)!(n-j)!}{(n)(n-1)\cdots(n-j+1)(n-j)!} \\
&= \frac{(j+1)(j)(j-1)\cdots(2)(1)}{(n)(n-1)\cdots(n-j+1)} \\
&= \left(\frac{j+1}{n}\right)\left(\frac{j}{n-1}\right)\cdots\left(\frac{2}{n-j+1}\right)
\end{aligned}$$

Since $0 < j \le n-1$, we have $0 < (j+c) \le (n-1+c)$ for $(1-j) \le c$. For such a c we have $(j+c)/(n-1+c) \le 1$ and hence,

$$\frac{(j+1)!(n-j)!}{n!} = \left(\frac{j+1}{n}\right)\left(\frac{j}{n-1}\right)\cdots\left(\frac{2}{n-j+1}\right) \le (1)(1)\cdots(1) \le 1$$

Alternate solution to part (a) Part (a) can also be shown using induction. Again note that this equality is obviously true for $j = 0$. For rest we use induction. For $n = 1$ we have $1! \le 1!$. Now let this be true for n, that is, $(k+1)!(n-k)! \le n!$ for $0 \le k \le n-1$.
Consider $1 \le j \le n$. Let $k = j-1$ and hence $0 \le k \le n-1$ Then,

$$\begin{aligned}
(j+1)!(n+1-j)! &= (j+1)(k+1)!(n+1-k-1)! \\
&= (j+1)(k+1)!(n-k)! \\
&\le (j+1)n! \\
&\le (n+1)n! \\
&= (n+1)!
\end{aligned}$$

(b) Since $x \in [0,1]$ we have $x_j \le x \le x_{j+1}$ for some integer $0 \le j \le n-1$. Using (1) we have

$$\prod_{i=0}^{n} |x - x_i| = \prod_{i=0}^{j-1} |x - x_i| \prod_{i=j}^{j+1} |x - x_i| \prod_{i=j+2}^{n} |x - x_i| \le \frac{h^2}{4} \prod_{i=0}^{j-1} |x - x_i| \prod_{i=j+2}^{n} |x - x_i|$$

From $x_j \le x \le x_{j+1}$, we have

$$|x - x_i| \le \begin{cases} (x_{j+1} - x_i), & i < j \\ (x_i - x_j), & i > j+1 \end{cases} \tag{2}$$

This implies,

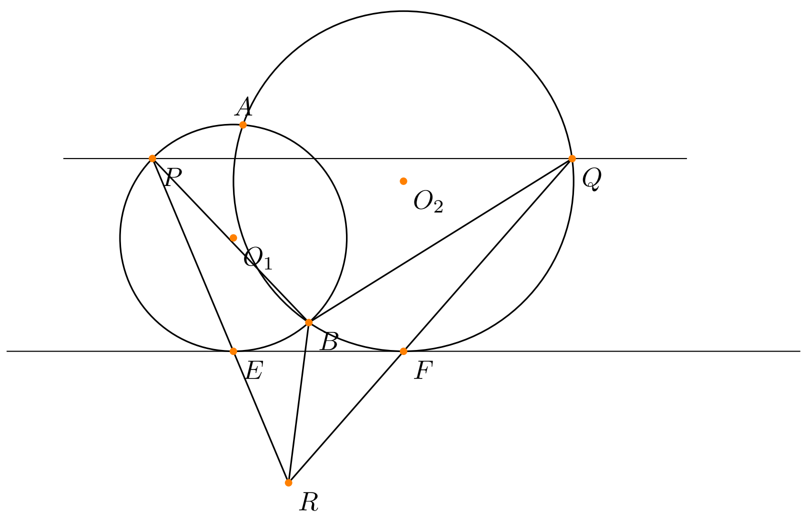
$$\begin{aligned}
\prod_{i=0}^{n}|x-x_i| &\le \frac{h^2}{4}\prod_{i=0}^{j-1}(x_{j+1}-x_i)\prod_{i=j+2}^{n}(x_i-x_j) \\
&\le \frac{h^2}{4}h^j h^{n-(j+2)+1}\prod_{i=0}^{j-1}(j-i+1)\prod_{i=j+2}^{n}(i-j)
\end{aligned}$$

where we used the fact that $x_{j+1} - x_i = (j+1)h - ih = (j-1+1)h$ and $x_i - x_j = (i-j)h$. Simplifying further,

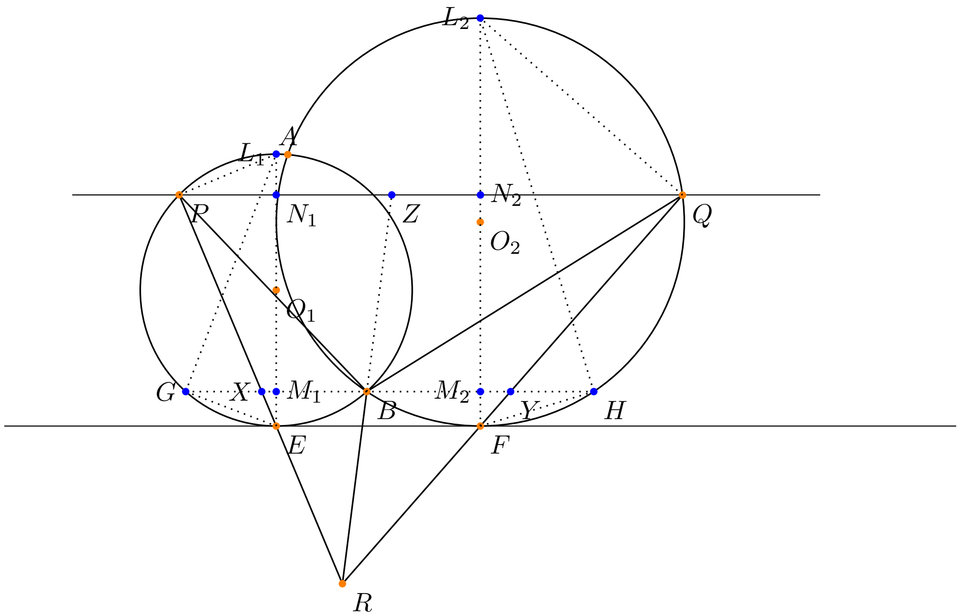
$$\prod_{i=0}^{n}|x-x_i| \le \frac{h^{n+1}}{4}(j+1)!(n-j)!$$

Finally, use the result from part(a) to finish the proof.

10. Here we have two circles C_1, C_2 with centers at O_1, O_2. C_1, C_2 intersect at two points A, B. P, E are two points in C_1; Q, F are two points in C_2. The line EF is a tangent line of C_1 and C_2. The line PQ is parallel to the line EF. The line PE intersects the line QF at the point R. Show that $\angle PBR = \angle QBR$.



Proof: Let us make a parallel line of EF passing through the point B. Suppose this line intersects the circle O_1 at G, the circle O_2 at H, the line PE at X, the line FQ at Y, the line EO_1 at M_1 and the line FO_2 at M_2. Let Z be the intersection point of the line BR and PQ, N_1 be the intersection point of EO_1 and PQ and N_2 be the intersection point of FO_2 and PQ. Let EO_1 intersects the circle O_1 at L_1 and FO_2 intersects the circle O_2 at L_2.



Let the radius of C_1 be r_1 and the radius of C_2 be r_2. Let the length of the line segment $EM_1 = FM_2$ be a and the length of the line segment $EN_1 = FN_2$ be b.

Notice that the line EF is tangent to the circle C_1, hence L_1E is perpendicular to the line EF. By our assumptions, GH is parallel to EF, thus the angle $\angle GM_1E$ is a right angle. Also the triangle $\triangle GEL_1$ is a triangle inscribed in a circle with one of the sides being the diameter. Thus the angle $\angle EGL_1$ is a right triangle with the hypothenuse being the diameter.

Since the triangle $\triangle GEM_1$ is similar to the triangle $\triangle GEL_1$, we have

$$|GE|^2 = |EM_1||EL_1| = 2ar_1.$$

Similarly since the triangle $\triangle EPN_1$ is similar to the triangle $\triangle EPL_1$, we have

$$|EP|^2 = |EN_1||EL_1| = 2br_1.$$

Notice that

$$\frac{|EX|}{|EP|} = \frac{a}{b},$$

we have $|EX| = \frac{a}{b}\sqrt{2br_1}$. Since the triangle $\triangle BXP$ is similar to the triangle $\triangle EXG$, we have

$$\frac{|BX|}{|BP|} = \frac{|EX|}{|EG|} = \sqrt{\frac{a}{b}}.$$

Similarly we have

$$\frac{|BY|}{|BQ|} = \sqrt{\frac{a}{b}}.$$

Thus

$$\frac{|BX|}{|BP|} = \frac{|BY|}{|BQ|}.$$

So we obtain

$$\frac{|BX|}{|BY|} = \frac{|BP|}{|BQ|}.$$

Notice that the line GH is parallel to the line PQ, thus we have

$$\frac{|XB|}{|PZ|} = \frac{|RB|}{|RZ|} = \frac{|BY|}{|ZQ|}.$$

That implies

$$\frac{|BX|}{|BY|} = \frac{|PZ|}{|ZQ|}.$$

Together, we have

$$\frac{|BP|}{|BQ|} = \frac{|PZ|}{|ZQ|}.$$

That shows $\angle PBZ = \angle QBZ$ which implies $\angle PBR = \angle QBR$.