UNM - PNM STATEWIDE MATHEMATICS CONTEST LII

November 1-4, 2019 First Round Three Hours

Note: Please provide **exact** answers to problems. E.g. if the answer to a problem is 1/3, **do not** approximate it as 0.33333.

1. Hussain is playing a game with his dad. The rule of the game is: in the n^{th} turn ($n \geq 1$), he has to roll a die n times. When he rolls a die, he will get a random number between 1 and 6. So in the n^{th} turn, he will get n numbers. His points a_n in the n^{th} turn is the sum of those numbers. He wins in the n^{th} turn if $a_n > 2^n$. What is the maximum number of turns that Hussain could possibly win?

2. Judy and Sarah bought a gift for their friend Kiera and placed it in one of 65 numbered boxes. They each tell Kiera that she is only allowed to open one box, but they each give her a clue. Judy says, the gift is in a box whose number has remainder 3 when divided by 5. Sarah tells Kiera that she might want to look in a box whose number is 7 more than a multiple of thirteen. Which box should Kiera open so that she finds the gift?

3. We have three robots, A, B and C, each of which is either an Autobot or a Decepticon. An Autobot always tells the truth while a Decepticon always lies.
A says: "C is a Decepticon".
C says: "A and B are of the same type (both Autobots or both Decepticons)."
Is B an Autobot or a Decepticon?

4. Everyday Manuel takes the escalator to his office located on the tenth floor. In the mornings, he is relaxed and goes up the escalator at the rate of one step per second, and ten steps brings him to the tenth floor. The afternoons are more frenzied, and Manuel goes up the escalator at four steps per second, reaching the tenth floor in thirty-six steps. How many steps are there in the escalator?

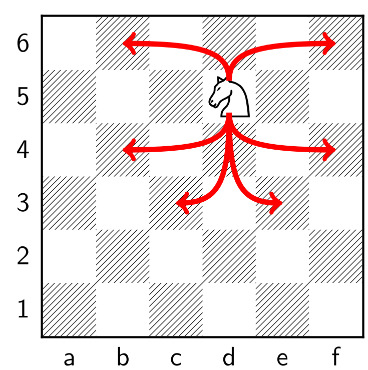
5. Find the total numbers of distinct pairs (x, y) such that x and y are integers and

$$x^2 + y^2 \leq 2x + 2y.$$

6. Given a positive integer $n \geq 33$, determine positive integers $x_1, x_2, \ldots, x_n$ such that

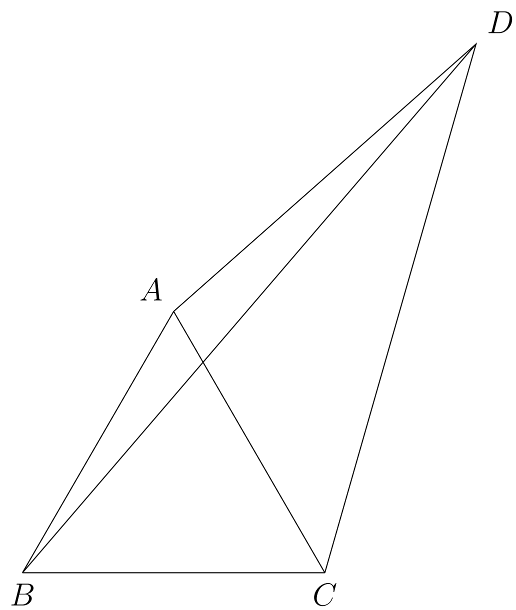
$$x_1 + 2(x_1 + x_2) + \cdots + n(x_1 + x_2 + \cdots + x_n) = \frac{2n^3 + 3n^2 + 13n - 6}{6}$$

7. A knight may move to a square that is two squares away horizontally and one square vertically, or two squares vertically and one square horizontally (but it can't move outside the chessboard). For example, the allowed moves for a knight are shown by arrows below.



We say a square is attacked by the knight if it can move to that square in its next turn. A knight is placed **randomly** on a square of a $\mathbf{6} \times \mathbf{6}$ chessboard. Then, a king is also placed **randomly** on one of the remaining squares. What is the probability that the king is on the square attacked by the knight?

8. Let $ABCD$ be a quadrilateral as shown below. Suppose that $\triangle ABC$ is an equilateral triangle, $\angle ADC = 30°$, the length of the side AD is 3 and the length of the side BD is 5. Caculate the length of the side CD.



9. Let p be an odd prime number, find out all the positive integers k such that $\sqrt{k^2 - pk}$ is a positive integer.
10. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers with $a_0 = 3$ and $(3 - a_{n+1})(6 + a_n) = 18$ for all natural number n. Calculate $\sum_{i=0}^{2019} \frac{1}{a_i}$.