UNM - PNM STATEWIDE MATHEMATICS CONTEST

November 1-4, 2019 First Round Three Hours

Note: Please provide **exact** answers to problems. E.g. if the answer to a problem is 1/3, **do not** approximate it as 0.33333.

1. Hussain is playing a game with his dad. The rule of the game is: in the n^{th} turn ($n \geq 1$), he has to roll a die n times. When he rolls a die, he will get a random number between 1 and 6. So in the n^{th} turn, he will get n numbers. His points a_n in the n^{th} turn is the sum of those numbers. He wins in the n^{th} turn if $a_n > 2^n$. What is the maximum number of turns that Hussain could possibly win?

   Answer: 4.

   Solution: In the n^{th} turn, $a_n \leq 6n$. Hussain is possible to win the n^{th} turn if $6n > 2^n$. That only happens if $n = 1, 2, 3, 4$. So the answer is 4.

2. Judy and Sarah bought a gift for their friend Kiera and placed it in one of 65 numbered boxes. They each tell Kiera that she is only allowed to open one box, but they each give her a clue. Judy says, the gift is in a box whose number has remainder 3 when divided by 5. Sarah tells Kiera that she might want to look in a box whose number is 7 more than a multiple of thirteen. Which box should Kiera open so that she finds the gift?

   Answer: 33.

   Solution: Let b be the box that Kiera should open. We have $b \ mod\ 5 = 3$ and $b \ mod\ 13 = 7$. Since $65 = 5 \cdot 13$, we can either use the Chinese Remainder Theorem, or, as the numbers are small, just search. The numbers less than 65 that give a remainder of 7 when divided by 13 are
   7, 20, 33, 46, 59. The only one of those that has remainder 3 upon division by 5 is 33.

3. We have three robots, A, B and C, each of which is either an Autobot or a Decepticon. An Autobot always tells the truth while a Decepticon always lies.
   A says: "C is a Decepticon".
   C says: "A and B are of the same type (both Autobots or both Decepticons)."
   Is B an Autobot or a Decepticon?

Answer: Decepticon

Solution: A is either an Autobot or a Decepticon. If A is an Autobot, then C is a Decepticon. Since C lies, that means B cannot be an Autobot, and so B is a Decepticon. Now, if A is a Decepticon, then C is an Autobot. Since C now tells the truth, B must be of the same type as A, that is, a Decepticon. In either case, B is a Decepticon.

4. Everyday Manuel takes the escalator to his office located on the tenth floor. In the mornings, he is relaxed and goes up the escalator at the rate of one step per second, and ten steps brings him to the tenth floor. The afternoons are more frenzied, and Manuel goes up the escalator at four steps per second, reaching the tenth floor in thirty-six steps. How many steps are there in the escalator?

   Solution: Let there be n total steps and the speed of escalator be v (in steps per second). In the morning, the escalator has to transport $n-10$ steps in 10 seconds. In the afternoon, the escalator has to transport $n-36$ steps in 9 seconds. So we have two equations,

$$n - 10 = 10v$$
$$n - 36 = 9v$$

   Solving gives $n = 270$.

5. Find the total numbers of distinct pairs (x, y) such that x and y are integers and

$$x^2 + y^2 \leq 2x + 2y.$$

   Answer: 9.

   Solution: Notice

$$x^2 + y^2 \leq 2x + 2y$$

   implies

$$x^2 - 2x + y^2 - 2y \leq 0.$$

   Adding 2 at both sides, we have

$$x^2 - 2x + 1 + y^2 - 2y + 1 \leq 2.$$

   That is

$$(x-1)^2 + (y-1)^2 \leq 2.$$

   Since $x-1$ and $y-1$ are integers, the only possible values for $x-1$ and $y-1$ are $-1, 0, 1$. So x, y can be $0, 1, 2$. There are nine choices totally.

6. Given a positive integer $n \geq 33$, determine positive integers $x_1, x_2, \ldots, x_n$ such that

$$x_1 + 2(x_1 + x_2) + \cdots + n(x_1 + x_2 + \cdots + x_n) = \frac{2n^3 + 3n^2 + 13n - 6}{6}$$

Answer: $x_i = 1$ for all i except $i = (n-1)$, and $x_{n-1} = 2$.

Solution:
Let
$$A = \frac{2n^3 + 3n^2 + 13n - 6}{6}.$$
The sum of squares formula gives us,
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + 3n^2 + n}{6}$$

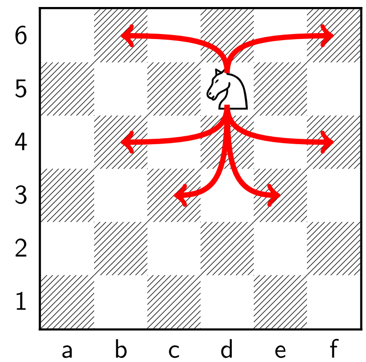
Setting all x_i's to 1 in $\{x_1 + 2(x_1 + x_2) + \cdots + n(x_1 + x_2 + \cdots + x_n)\}$ (which is their minimum possible value) and then using the sum of squares formula gives $\frac{2n^3 + 3n^2 + n}{6}$. The difference between this value and A is,
$$\frac{2n^3 + 3n^2 + 13n - 6}{6} - \frac{2n^3 + 3n^2 + n}{6} = \frac{12n - 6}{6} = 2n - 1,$$

So, we have to increase some x_i's to make up this difference of $2n - 1$. If we increase x_1 to 2, the net increase is $1 + 2 + \cdots n = n(n+1)/2 > 2n - 1$ for $n \geq 33$. In fact, the equality holds only for $n = 2 < 33$, while for any $n > 3$, we have strictly inequality.

However, this gives us an idea to increase x_{n-1} to 2. x_{n-1} is present in only the last two terms, so that increases the sum by $n - 1 + n = 2n - 1$. So, $x_i = 1$ for all i except $i = (n-1)$, and $x_{n-1} = 2$ is a solution. We can also check that this is a unique solution. Increasing any x_i for $i < n - 1$ will lead to an increase larger than $2n - 1$, since that x_i will be present not only in the last two terms in parenthesis (which contribute $2n - 1$) but also in at least one other term in parenthesis. Increasing x_n to 2 still falls short by $n - 1$ and if we follow this up by increasing any other x_i, we exceed our desired sum. Setting $x_n = 3$ increases the sum by $2n$ which is now more than $2n - 1$. So, we have only one solution that we found above.

7. A knight may move to a square that is two squares away horizontally and one square vertically, or two squares vertically and one

square horizontally (but it can't move outside the chessboard). For example, the allowed moves for a knight are shown by arrows below.



We say a square is attacked by the knight if it can move to that square in its next turn. A knight is placed **randomly** on a square of a $\mathbf{6} \times \mathbf{6}$ chessboard. Then, a king is also placed **randomly** on one of the remaining squares. What is the probability that the king is on the square attacked by the knight?

Answer: $160/1260 = 8/63$

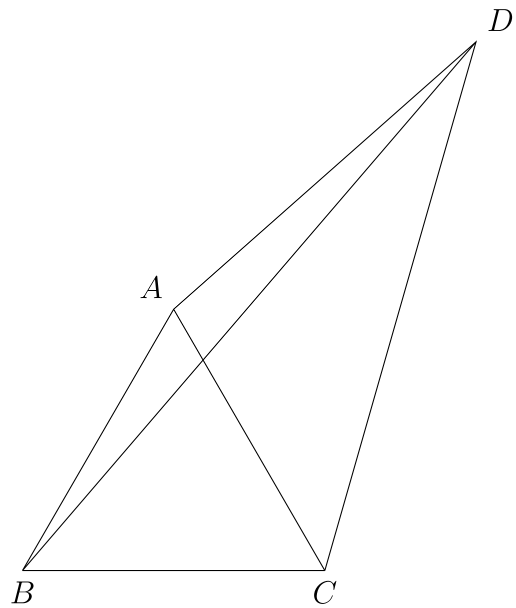
Solution: We can label the squares of the chessboard based on how many other squares the knight attacks from that square. For example, if the knight is placed on the upper left corner it attacks two other squares. The figure below shows the number of other squares attacked by a knight when it is on a particular square.

2	3	4	4	3	2
3	4	6	6	4	3
4	6	8	8	6	4
4	6	8	8	6	4
3	4	6	6	4	3
2	3	4	4	3	2

Let P be the desired probability. The knight can land on any square with probability $1/36$. The king can then land on any of the the remaining squares with probability $1/35$. P is the sum of probabilities that a knight placed on a certain square attacks the king placed randomly on one of the other squares. Or,

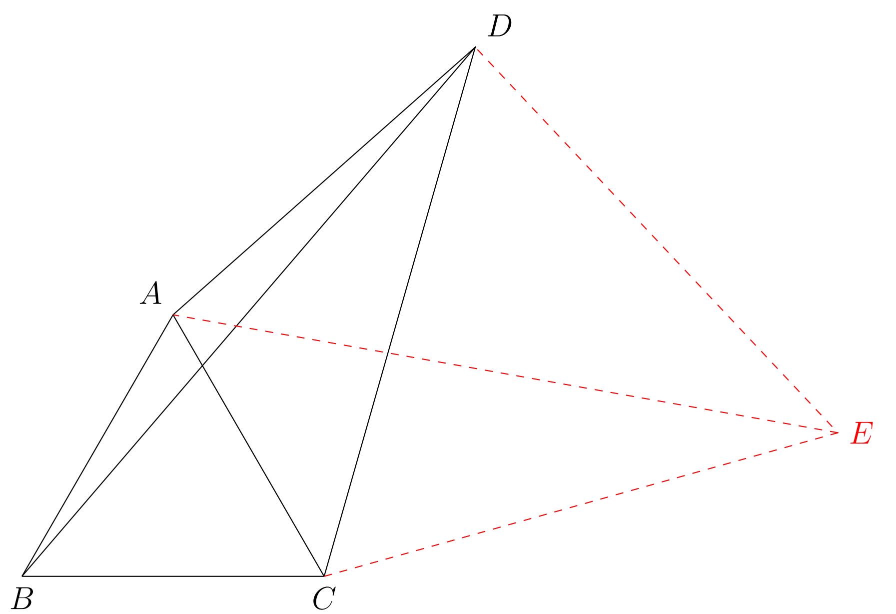
$$P = (4/36) * (2/35) + (8/36) * (3/35) + (12/36) * (4/35) + (8/36) * (6/35) + (4/36) * (8/35)$$
$$= 160/1260 = 8/63$$

8. Let $ABCD$ be a quadrilateral as shown below. Suppose that $\triangle ABC$ is an equilateral triangle, $\angle ADC = 30^\circ$, the length of the side AD is 3 and the length of the side BD is 5. Caculate the length of the side CD.



Answer: 4.

Solution: Let us make an equilateral triangle $\triangle CDE$. We claim that the two triangle $\triangle BCD$ and $\triangle ACE$ are congruent triangles.



Notice that $\angle ACB = \angle DCE = 60^\circ$, we have $\angle DCB = \angle ACE$. From our assumptions we have $|AC| = |CB|$, $|DC| = |CE|$, thus $\triangle DCB$ and $\triangle ACE$ are congruent. As a consequence, we have $|AE| = |BD| = 5$. Since $\angle ADE = \angle ADC + \angle CDE = 30^\circ + 60^\circ = 90^\circ$, we have $|DE| = 4$. Thus $|CD| = 4$.

9. Let p be an odd prime number, find out all the positive integers k such that $\sqrt{k^2 - pk}$ is a positive integer.

Answer: $k = \left(\frac{p+1}{2}\right)^2$.

Solution: Since p is an odd prime number, the greatest common divisior of $k - p$ and k is 1 unless $k = \lambda p$, where λ is a positive integer. Suppose $k = \lambda p$, then $k^2 - pk = (k - p)k = (\lambda - 1)\lambda p^2$. It is impossible to find $\lambda > 1$ such that $(\lambda - 1)\lambda$ is a square number. Otherwise since $gcd(\lambda - 1, \lambda) = 1$, λ and $\lambda - 1$ are all square numbers which is not possible.

Now we consider the case that $gcd(k - p, k) = 1$. Since $(k - p)k$ is a square number, we conclude that k and $k - p$ are all square numbers. Let $k = a^2$, $k - p = b^2$, then $p = a^2 - b^2 = (a - b)(a + b)$. Thus $a - b = 1$. It implies that $a + b = p$. So $a = \frac{p+1}{2}$ and $k = \left(\frac{p+1}{2}\right)^2$.

10. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers with $a_0 = 3$ and $(3 - a_{n+1})(6 + a_n) = 18$ for all natural number n. Calculate $\sum_{i=0}^{2019} \frac{1}{a_i}$.

Answer: $\sum_{i=0}^{2019} \frac{1}{a_i} = \frac{2^{2021} - 2022}{3}$.

Solution: From $(3 - a_{n+1})(6 + a_n) = 18$, we have $3 - a_{n+1} = \frac{18}{6 + a_n}$. Thus $a_{n+1} = 3 - \frac{18}{6+a_n} = \frac{3a_n}{6+a_n}$. So $\frac{1}{a_{n+1}} = \frac{6+a_n}{3a_n} = \frac{2}{a_n} + \frac{1}{3}$. If we add $\frac{1}{3}$ to both sides, we obtain

$$\frac{1}{a_{n+1}} + \frac{1}{3} = 2\left(\frac{1}{a_n} + \frac{1}{3}\right).$$

Hence $\frac{1}{a_n} + \frac{1}{3}$ is a geometric sequence with common ratio 2. It implies that $\frac{1}{a_n} + \frac{1}{3} = 2^n\left(\frac{1}{a_0} + \frac{1}{3}\right) = \frac{2^{n+1}}{3}$. Thus

$$\begin{aligned} &\sum_{n=0}^{2019}\left(\frac{1}{a_n} + \frac{1}{3}\right) \\ =&\sum_{n=0}^{2019}\frac{2^{n+1}}{3} \\ =&\frac{2^{2021} - 2}{3} \end{aligned}$$

Thus we have $\sum_{i=0}^{2019} \frac{1}{a_i} = \frac{2^{2021}-2022}{3}$.