UNM – PNM STATEWIDE MATHEMATICS CONTEST LIII

February 6, 2021 Second Round Three Hours

Instructions

Please read the following instructions carefully before you start working on the problems.

- There are a total of 10 questions in this exam.
- Write your complete solution for each problem. Remember to show all work necessary to justify your answer. If you are asked for a numerical answer, please circle your final answer. Partial credit will be given for clearly written contributions toward a solution.
- No calculators allowed.
- You may write your solution on paper, or use an electronic device like a tablet. In either case, you will need to convert your solution to a PDF. The instructions for converting solutions written on paper are given next. If you use some electronic device (e.g. iPad) for your solutions, it's your responsibility to know how to convert your solutions into a single PDF file.
- If you wrote the exam solutions on sheets of paper, follow the instructions given by Gradescope to create a PDF file of your work: `https://gradescope-static-assets.s3.amazonaws.com/help/submitting_hw_guide.pdf`.
- Click on "SUBMIT PDF" and submit your work, following Gradescope's instructional video: `https://www.youtube.com/watch?v=u-pK4GzpId0`.
- Note that you will then need to label which pages of your submission correspond to which problems on the exam!
- The following notation is used:
  - Given a positive integer n we define $n! = (n)(n-1)(n-2)\cdots(2)(1)$, and define $0! = 1$.

Questions

1. Mr. Candelaria has three kids. They were all born in January but in different years. Mr. Candelaria's age is a multiple of the sum of kids' age, and also a multiple of the product of kids' age. The youngest kid's age is not less than 2 years, while Mr. Candelaria's age is not greater than 50 years.

   (a) What are the possible ages of kids? (Note: age must be an integer).
   (b) What is the probability that kids' birthdays are all on Saturday in 2021? (Note: The 1^{st} of January 2021 was a Friday, and January has a total of 31 days).

2. Sophia, a bright high school student, decided to experimentally estimate the height of the highest point on the Sandia Mountains. On a clear morning, while looking at the Sandias, Sophia could see the sun peeking above the peak. Sophia's height is h meters, and, with the sun still in the same place, she measured her own shadow on the ground to be s meters.

   Sophia was wearing a helmet with a laser mounted on top. After turning on the laser, the emitted beam reached the peak of the Sandias in t seconds, as measured by an accurate experimental setup she had previously devised for a science fair. After taking into account her experimental findings and assuming the laser beam traveled at a speed of c million meters per second, what height did Sophia estimate for the peak of the Sandias?

3. A computer scientist is writing an iPhone app in which she needs to evaluate the following expression:

$$\log \sum_{i=1}^{2021} e^{x_i},$$

   where x_i are real numbers. However, the computer scientist notes that evaluating e^{x_i} for x_i larger than 88 results in the app crashing. How could she reformulate the above expression so that the new expression is mathematically the same, but computing it doesn't crash the app?

4. How many distinct squares can be drawn on a grid 11 cells wide and 13 cells long, assuming each square's corners must be on a grid point?

5. For the set of positive real numbers $X = \{x_1, \ldots, x_n\}$, let

$$s_k = x_1^k + \cdots + x_n^k,$$

   and p_k be the sum of all their possible products taken k at a time. Prove that

$$(n-1)!s_k \geq k!(n-k)!p_k.$$

   (Example of a special case: Let $X = \{1, 2, 4\}$, and therefore $n = 3$. For $k = 2$, $s_k = 1^2+2^2+4^2 = 21$ and $p_k = (1)(2) + (1)(4) + (2)(4) = 14$. In this case, $(3-1)!21 = 42 \geq (2!)(3-2)!14 = 28$.)

6. The integer part of a real number x is the greatest integer less than or equal to x, and is denoted by $\lfloor x \rfloor$. For example, $\lfloor 4.5 \rfloor = 4$ and $\lfloor -4.5 \rfloor = -5$. Show that all non-negative solutions to the equation

$$x = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{4} \right\rfloor + \left\lfloor \frac{x}{5} \right\rfloor + \left\lfloor \frac{x}{6} \right\rfloor$$

   are $x = 0, 4, 5$.

7. A triangle ABC is such that the side BC has length 20 (units). Line XY is drawn parallel to BC such that X is on segment AB and Y is a point inside the triangle. The line XY (when extended) intersects CA at the point Z. Line BZ bisects angle YZC. If XZ has length 12 (units), then what is the length of AZ?

8. Find all polynomials $p(x)$ satisfying the identity $(x-1)p(x+1) = (x+2)p(x)$ for all real numbers x.

9. On New Year's Eve, yet another driver from Texas was seen speeding along I-40 near Santa Rosa. The license plate number is a four-digit number equal to the square of the sum of the two two-digit numbers formed by taking the first two digits and the last two digits of the license plate number. It's also known that the first digit of the license plate is not zero. What is the four-digit license plate number?

10. In a round-robin tournament, n teams are put into $n/2$ pairs. Each pair of teams plays a game in round 1, and the winner moves to the next round. In round 2, the $n/2$ teams are paired, and the winners move on to round 3, which will have $n/4$ teams. This continues until the final round has two teams, and the winner of this final game wins the tournament. In a round-robin tournament with 2^n teams, there are n rounds.

    Suppose a round-robin sports tournament has 3 teams from Albuquerque and 13 teams from elsewhere in New Mexico, for a total of $16 = 2^4$ teams. Assume that each team is equally skilled so that each team has a 50% chance of winning each game; that all games are independent; and each game results in one team winning, so that there are no ties.

    (a) Assuming that the initial pairings of the teams is done at random, and therefore all pairings are equally likely, what is the probability that one of the teams from Albuquerque wins the tournament?

    (b) Assuming that the initial pairings of the teams is done at random, and therefore all pairings are equally likely, find the probability that in the first round, two of the teams from Albuquerque play each other.

    (c) Instead of pairing teams at random, suppose that in round 1, two of the Albuquerque teams are paired, while the third Albuquerque team is paired with another team at random. This guarantees that at least one Albuquerque team wins and at least one Albuquerque team loses in the first round. Also suppose that if there are two Albuquerque teams available in round 2, they are planned to play each other. Thus in round three there are either 0 or 1 Albuquerque teams. Under this setting, find the probability that an Albuquerque team wins the tournament.