UNM - PNM STATEWIDE MATHEMATICS CONTEST

October 31 - November 2, 2020 First Round Three Hours

1. Twenty lemons cost the same number of coins as the number of lemons that you could buy for 500 coins. How much do 10 lemons cost?

   Answer: 50

   Solution: Denote the cost (in coins) of one lemon by c. Then $20c = 500/c$, from which $c = 5$. The cost of 10 lemons is therefore $10c = 50$ coins.

2. The lengths (in centimeters) of of each of four triangles I, II, III and IV are as follows:

   $$I : 20, 21, 29 \qquad II : 3, 4, 5$$
   $$III : 4, 7\frac{1}{2}, 8\frac{1}{2} \qquad IV : 7, 8\frac{1}{2}, 11\frac{1}{2}$$

   Which of these are right-triangles (that is, triangles having a right angle)?

   Answer: I,II and III.

   Solution: The sum of squares of the lengths of two sides of these triangles equals to the square of the length of the third side.

3. A jar contains 2020 balls. Each ball may have one or more colors on it (for example, a ball could be completely colored red, another could be colored red and blue, and another could be colored green, yellow and orange). The only information we have is that the number of balls with red on them (for example, a ball which is colored in both blue and red) is a multiple of 7. Similarly, the number of balls with blue on them is a multiple of 9, and the number of balls with yellow on them is a multiple of 15. What is the maximum number of balls colored both red and blue but not yellow?

   Answer: 2016

   Solution: The maximum number of balls having both red and blue must be a multiple of $\mathrm{lcm}(7, 9) = 63$. We have $32 \cdot 63 = 2016$. The balls having yellow play no role in the maximum since that number could be 0.

4. 90 students attended the Public Lecture of the statewide UNM-PNM math contest last year. Some of these students arrived early and had snacks which were served before the lecture started. Donuts were eaten by 32 students, cupcakes were eaten by 61 students, and sandwiches were eaten by 29 students. 28 students had both a donut and a cupcake, 16 students had both a cupcake and a sandwich, and 10 students had both a donut and a sandwich. Only 6 students ate all three snacks. How many students did not eat any snack?

   (Note: The online version had a typo and stated that sandwiches were eaten by 19 students. We gave everyone full credit for this problem.)

   Answer: 16

   Solution: Let D, S, and C be the set of students who ate donuts, cupcakes, and sandwiches respectively. Then, $|D| = 32, |C| = 61, |S| = 29, |D \cap C| = 28, |D \cap S| = 10, |C \cap S| = 16, |D \cap S \cap C| = 6$ (where $|A|$ denotes the number of elements in the set A). Then,

$$\begin{aligned} |D \cup S \cup C| &= |D| + |C| + |S| - |D \cap C| - |D \cap S| - |C \cap S| + |D \cap S \cap C| \\ &= 32 + 61 + 29 - 28 - 10 - 16 + 6 = 74. \end{aligned}$$

   Therefore, students who did not have any snack are $90 - 74 = 16$.

5. Suppose $x_1 = a$ and $x_{n+1} = x_n + b$. For what values of a, b do we have $x_0 + \cdots x_n = n^2$?

   Answer: $a = 1$, $b = 2$.

   Solution: We have

$$x_0 + \cdots x_n = \frac{n(2a + (n-1)b)}{2} = \left(a - \frac{b}{2}\right) n + \frac{b}{2} n^2.$$

   In order for this sum to equal n^2 for all n, we must have that

$$\begin{aligned} a - \frac{b}{2} &= 0, \\ \frac{b}{2} &= 1, \end{aligned}$$

   from which $b = 2$ and $a = 1$.

   A different solution by Bill Cordwell: Since $x_1 = 1 = 1^2$, we must have $a = 1$. The perfect squares are obtained by successively adding the consecutive odd integers (note $(n+1)^2 - n^2 = 2n + 1$), e.g., 1, $1 + 3 = 4$, $4 + 5 = 9$, $9 + 7 = 16$, etc., so since each successive odd number is 2 more than the previous ($x_{n+1} = x_n + 2$, $b = 2$. So, $a = 1$ and $b = 2$.

6. What are the last two digits of 6^{2020}?

   Answer: 76

   Solution: We want 6^{2020} $mod 100$. Note that,

$$
\begin{aligned}
6^1 &= 6 \\
6^2 &= 36 \\
6^3 &= 16 \ mod \ 100 \\
6^4 &= 96 \ mod \ 100 \\
6^5 &= 76 \ mod \ 100 \\
6^6 &= 56 \ mod \ 100 \\
6^7 &= 36 \ mod \ 100
\end{aligned}
$$

   For $n \geq 2$, $6^n = 6^{5k+r}$ mod 100, where r is the remainder mod 5. Now $2020 = 0$ mod 5, and hence $6^{2020} = 76$.

7. Four excellent mathematics school students are walking down the street in downtown Gallup, NM, when they see an absolutely flagrant violation of the New Mexico Rules of the Road. As the car is speeding away, the first student notes the license plate starts with "TX" and is followed by four digits; the second student notes the next two digits are identical to each other; the third student notes the *last* two digits are identical to each other; and the final student notes the four digits form a perfect square. What is the license plate number?

   Answer: TX7744

   Solution: The four digits of the plate form the number

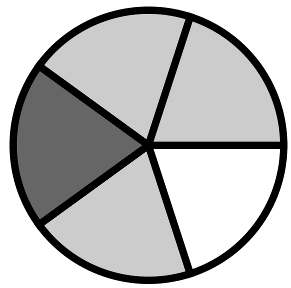
$$n^2 = 1000a + 100a + 10b + b = 11(100a + b).$$

   Because 11 divides the perfect square n^2, it must be that 11^2 divides it, too; and therefore 11 divides $100a + b$. Either by the divisiblity rule for 11 or by considering this sum modulo 11, we see that 11 divides $a + b$. Because a and b are single digit numbers, it must be that $a + b = 11$.

   Because n^2 is perfect square, its last digit—and therefore b—must be one of $0, 1, 4, 5, 6, 9$, to which the corresponding possible values for a are $11, 10, 7, 6, 5, 2$. Because a and b are single digit numbers, the only possible license plate numbers are 7744, 6655, 5566 and 2299. Of these, only $7744 = 88^2$ is a perfect square.

8. A disc is divided into p sectors, where p is a prime. Each sector is painted using one of n colors. How many ways are there to paint the disc if $n = 5$ and $p = 7$? Count all configurations which are rotations of each other as one case.

   For reference, below is an example of a disc with 5 sectors and 3 colorings (dark gray, light gray, white).



   Answer: 11165

   Solution: There are n^p ways to color the disc, of which n are solid coloring (i.e. each sector is colored the same). Because each non-solid coloring can be rotated into $p-1$ other orientations, there are

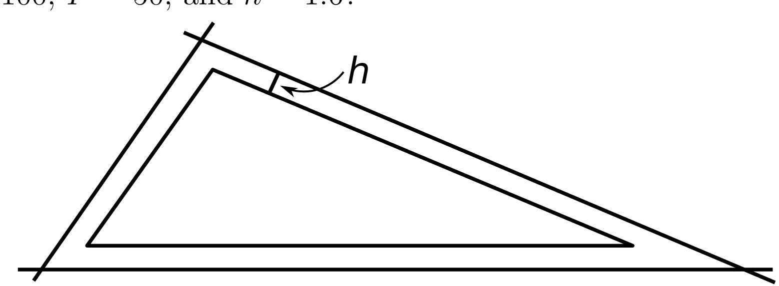
$$\frac{n^p - n}{p}$$

non-solid colorings, and

$$\frac{n^p - n}{p} + n$$

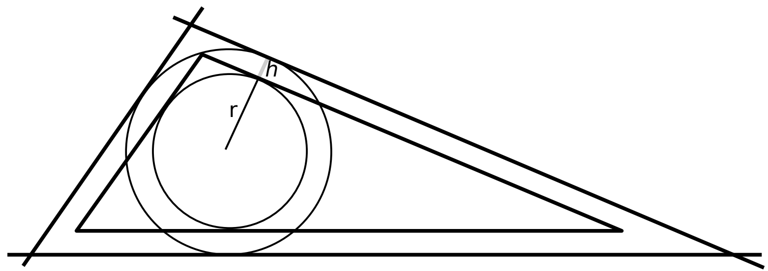
total colorings. Plugging in $n = 5$ and $p = 7$ into the above expression gives 11165. (Note the necessary calculation may be sped up by writing $5^7 = 5^3 5^4 = 125 \cdot 625$.)

9. A triangle has area S and perimeter P. Each of the lines forming the sides of the triangle are moved outward a distance h in the direction that's perpendicular to the corresponding side of the triangle, as in the figure below. What is the area S' of the new triangle when $S = 100$, $P = 50$, and $h = 1.6$?



Answer: 144

Solution: The new triangle is similar to the original triangle, and has the proportionality constant $\kappa = \frac{r+h}{r} = 1 + \frac{h}{r}$, where $r = \frac{2S}{P}$ is the radius of the circle inscribed in the triangle. We therefore have $\kappa = 1 + \frac{hP}{2S}$, from which $S' = \kappa^2 S = \left(1 + \frac{hP}{2S}\right)^2 S$. Plugging in the given values for S, P, h we get $S' = 1.96S = 196$.

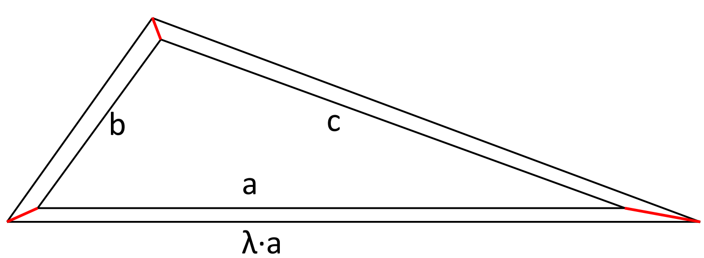


A different solution by Bill Cordwell: By construction, each of the expanded sides is parallel to its original side, so all of the angles of the expanded triangle are the same as the original triangle. Thus, the two triangles are similar.

Then each of the sides a, b, and c is magnified in length by a scale factor λ. This means the area is magnified by a factor of λ^2. If we draw lines (in red, above) connecting the nearby vertices, we see that the extra area added is the sum of three long, skinny trapezoids, with the bottom one adding $\frac{1}{2}(\lambda a + a) \cdot h$, for example. Adding all three trapezoids gives an area $A' = A + \frac{1}{2}(P' + P) \cdot h = A + \frac{1}{2}(\lambda P + P) \cdot h = \lambda^2 A$.

Substituting for A and P gives, simplifying, $5\lambda^2 - 2\lambda - 7 = 0$, and solving for λ gives the scale factor of 1.4. The new area is then $(1.4)^2 \cdot 100 = 196$.

This approach would also work if the h values were different for each side, but we would need to know the original lengths of the sides in that case.



10. Given a positive integer n, we know that the number of non-negative integer solutions to

$$x_1 + x_2 + \cdots x_k = n$$

is $\binom{n+k-1}{k-1}$, where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and $n! = n \times (n-1) \times (n-2) \cdots 2 \times 1$. For example consider finding non-negative integer solutions to,

$$x + y = 3.$$

Here $n = 3, k = 2$, and so we have $\binom{4}{1} = 4$ solutions (these are $x = 0, y = 3$, $x = 1, y = 2$, $x = 2, y = 1$, and $x = 3, y = 0$).

Now we want to find the number of non-negative integer solutions to

$$a + b + c + d = 16$$

such that $2 \leq a \leq 5, \quad 1 \leq b \leq 8, \quad 0 \leq c \leq 6, \quad 3 \leq d \leq 8$. How many such solutions are there?

Answer: 138

Solution: Substitute $w = a - 2, x = b - 1, y = c, z = d - 3$. Then we want the number of solutions to

$$w + x + y + z = 10 \qquad (*)$$

such that $0 \leq w \leq 3, 0 \leq x \leq 7, 0 \leq y \leq 6, 0 \leq z \leq 5$. Let S denote all the non-negative integer solutions to $(*)$, S_1 denote the subset such that $w \geq 4$, S_2 the subset such that $x \geq 8$, S_3 the subset such that $y \geq 7$ and S_4 the subset such that $z \geq 6$. From the statement of the problem we know that the number of solutions in S is $\binom{13}{3}$.

The number of solutions in S_1 are the same as the number of non-negative integral solutions to

$$w + x + y + z = 6$$

(obtained by subtracting 4 on both sides of $(*)$). This number is $\binom{9}{3}$. Similar, the number of solutions in S_2, S_3 and S_4 are $\binom{5}{3}$, $\binom{6}{3}$ and $\binom{7}{3}$ respectively.

Now consider the number of solutions in $S_1 \cap S_2$. These are non-negative integral solutions to $(*)$ such that $w \geq 4, x \geq 8$. This number is obviously 0. Now, number of solutions in $S_1 \cap S_3$ satisfy $w \geq 4, y \geq 7$. This number is also 0. For $S_1 \cap S_4$ we want $w \geq 4, z \geq 6$. This is the same as the number of non-negative integer solutions to

$$w + x + y + z = 0$$

This number is $\binom{3}{3} = 1$. Similarly, number of solutions in $S_2 \cap S_3$, $S_2 \cap S_4$, $S_3 \cap S_4$ are all zero. From this it also follows that number of solutions in intersection of three or more of these sets is also 0.

Let $|S|$ denote the number of elements of S. Then, our answer is

$$\begin{aligned} |S| - |S_1 \cup S_2 \cup S_3 \cup S_4| &= |S| - |S_1| - |S_3| - |S_3| - |S_4| + |S_1 \cap S_4| \\ &= \binom{13}{3} - \binom{9}{3} - \binom{5}{3} - \binom{6}{3} - \binom{7}{3} + 1 = 138 \end{aligned}$$

(The remaining terms are 0.)