UNM-PNM Solutions

1.
   - (a) How many distinct whole number factors does 6 have?
     Answer: 4
   - (b) How many distinct whole number factors does 12 have?
     Answer: 6
   - (c) How many distinct whole number factors does 144 have?
     Answer: 15
     Solution: $144 = 2^4 3^2$, so the number of factors is $(4+1)(2+1) = 15$.

2. When there's nothing better to do, Vitali and JoJo meet up to wash Ms. Miriam's cat, Clementina. Vitali and JoJo noticed that after washing Clementina thirty-five times, the bar of soap's width shrunk by a factor of 2, the height shrunk by a factor of 3, and the length shrunk by a factor of 6. How many more times can JoJo and Vitali wash Clementina, before having to open up a new bar of soap?

   Answer: 1

   Solution: Denote by f the fraction of a new bar of soap that's used to was Clementina once. Then

   $$35f = \left(1 - \frac{1}{2 \times 3 \times 6}\right),$$

   i.e. $f = 1/36$. Since just $1/36$ of the soap is left, JoJo and Vitali can wash Clementina just one more time!

3. In anticipation of the upcoming new year, four residents of San Jon sat down in the park and ate 2022 chiles. Each of the four people ate a different number of chiles. The two people who ate the second- and third- largest number of chiles ate 1011 chiles in total.

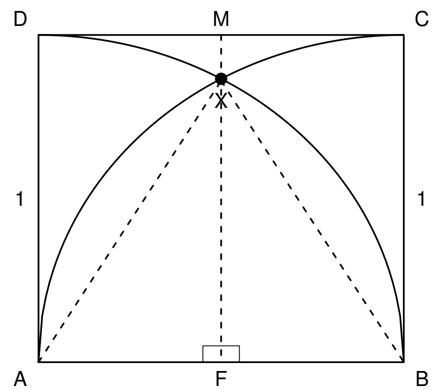
   - (a) How many chiles were eaten in total by the two people who ate the smallest and greatest number of chiles?
     Answer: 1011
   - (b) What is the greatest possible number of chiles that could have been eaten by the San Jonian who ate the fewest chiles?
     Answer: 504 chiles
     Solution: Note that $1011 = 504+505$. Thus, the most chiles the fourth resident could've eaten is 503, and the least chiles the first resident could've eaten is 506. Since $503+504+505+506 = 2022$, this is the most favorable allocation is chiles with respect to resident 4.

4. Inside unit square $ABCD$, quarter circle arcs with centers A and B are drawn. These arcs intersect at point X inside the square. If the distance from X to side AB is used as the side length of a square, what is its area?

   Answer: $3/4$ (assuming that the quarter circles have radius 1–Other answers were accepted if this was not assumed).

   Solution: Draw M and F as shown. The quarter circle arcs AXC and BXD intersecting at X have radii 1 (XB is the radius of a radius 1 circle centered at B), so that triangle AXB is equilateral. Thus, XF has length $\sqrt{3}/2$.

   The area of a square with this side length is $(\sqrt{3}/2)^2 = 3/4$.



5. Suppose $x, y, z > 0$ and $xyz = 1$. What is the difference between the maximum and minimum value the expression

$$\frac{1}{1+x+xy} + \frac{1}{1+y+yz} + \frac{1}{1+z+zx} \tag{1}$$

can attain?

Answer: 0

Solution:

$$\begin{aligned}
\frac{1}{1+x+xy} + \frac{1}{1+y+yz} + \frac{1}{1+z+zx} &= \frac{1}{1+x+xy} + \frac{1}{1+y+yz} + \frac{1}{1+z+zx} \\
&= \frac{1}{1+x+xy} + \frac{x}{x(1+y+yz)} + \frac{xy}{xy(1+z+zx)} \\
&= \frac{1}{1+x+xy} + \frac{x}{x+xy+xyz} + \frac{xy}{xy+xyz+xyzx} \\
&= \frac{1}{1+x+xy} + \frac{x}{x+xy+1} + \frac{xy}{xy+1+1x} \\
&= \frac{1+x+xy}{1+x+xy} \\
&= 1,
\end{aligned}$$

so the given sum is identically equal to 1! Thus, the answer is $1 - 1 = 0$.

6. A cube is inscribed in a sphere of radius 1. What is the surface area of the cube?

Solution: 8 (units^2)

Solution: Suppose the side length of the cube is a. Then the length of any side's diagonal is $\sqrt{2}a = \sqrt{a^2 + a^2}$, and the length of the cube's diagonal is therefore given by $\sqrt{2a^2 + a^2} = \sqrt{3}a$. However, since the length of the diagonal is twice the circumscribing sphere's radius, we have that $a = 2/\sqrt{3}$.

Since the surface area of a cube is given by $6a^2$, we get that the surface area is $6 \times \frac{2^2}{3} = 8$ area units.

7. A natural number between 1 and 2021 (inclusive) is chosen with uniform probability. What is the probability that it is a perfect power, i.e. can be written as m^k, where m, k are natural numbers satisfying $m \geq 1$, $k > 1$?

Answer: $\frac{56}{2021}$

Solution: Note that $x^{nm} = (x^n)^m$, so we only need to count how many numbers there are less than 2021 which are a prime power of a whole number. We can compute by hand that these are 1; $2^2, 3^2, \ldots, 44^2$; $2^3, 3^3, \ldots, 12^3$; $2^5, 3^5, 4^5$; and 2^7. Of these, $4^3 = (2^3)^2$, $9^3 = (3^3)^2$ and $4^5 = (2^5)^2$ show up twice. Thus, the total number of perfect powers is $1 + 43 + 11 + 3 + 1 - 3 = 56$, and so the probability of choosing a perfect power is

$$\frac{56}{2021}.$$

8. If you were to write down the numbers $1, 2, \ldots, 2021$ on a chalkboard, how many times would you write down the number 3?

Answer: 602

Solution: Let x be a placeholder for a digit not equal to 3, and y a placeholder for a digit not equal to 0 or 3. The numbers between 1 and 2021 which have a 3 in their digit expansion look as follows:

(a) 1 digit: 3 : 1 possibility

(b) 2 digits: $3x, y3, 33$: $9 + 8 = 17$ possibilities with one 3 digit, 1 possibility with two 3 digits

(c) 3 digits: $3xx, y3x, yx3, 33x, 3x3, y33, 333$: $9^2 + 8 \cdot 9 + 8 \cdot 9 = 225$ possibilities with one 3 digit; $9 + 9 + 8 = 26$ possibilities with two 3 digits; one possibility with all 3 digits equal to 3

(d) 4 digits, less than or equal to 2021: $13xx$, $1x3x$, $1xx3$, $133x$, $13x3$, $1x33$, 1333, 2003, 2013 : $3 \times 9^2 + 2 = 245$ possibilities with one 3 digit; $3 \times 9 = 27$ possibilities with two 3 digits; one possibility with three digits equal to 3

Thus, you wrote down $1(1 + 17 + 225245) + 2(1 + 26 + 27) + 3(1 + 1) = 602$.

9. Two people plan to meet at a particular location between 11:00 am and noon on a given day. They each arrive at a random time within the hour. They agree that if the first person there has to wait 15 minutes or more, that person will leave, and the two will not meet. What is the probability that they meet?

Answer: 7/16

Solution: Suppose the first person arrives at time X and the second at time Y. We can represent the combination of times they arrive as a point on the unit square, where 0 maps to 11:00am and 1 to noon. They arrive within 15 minutes of each other (and therefore meet) if $|X - Y| \leq 1/4$ (scaling X and Y to $[0, 1]$). This makes an irregular region within the unit square with area 7/16, which is the answer.

10. What is the product of the solutions to the equation $\left[\frac{2x-1}{3}\right] = \frac{x-1}{2}$? Here $[t]$ means the integer part of t, i.e., the greatest integer less than or equal to t. For example, $[-6.4] = -7$, $[6.4] = 6$, $[1] = 1$.

Answer: -3

Solution: Note that $t = (x - 1)/2$ is an integer. The equation can be rewritten as

$$\left[\frac{2(2t+1) - 1}{3}\right] = t \Leftrightarrow \left[\frac{4t+1}{3}\right] = t.$$

Note that

$$\left[\frac{4t+1}{3}\right] \leq \frac{4t+1}{3} < \left[\frac{4t+1}{3}\right] + 1,$$

or equivalently

$$0 \leq \frac{4t+1}{3} - \left[\frac{4t+1}{3}\right] < 1.$$

Using the fact that $\left[\frac{4t+1}{3}\right] = t$, the last inequality can be rewritten as

$$0 \leq \frac{t+1}{3} < 1 \Leftrightarrow -1 \leq t < 2,$$

from which $t = -1, 0, 1$ and therefore $x = -1, 1, 3$. The product of these numbers is $(-1)(1)(3) = -3$.