Math Contest 2022-2023 Round I Test with Solutions

1. Regular gas costs \$4 per gallon and premium gas costs \$4.20 per gallon. If a car gets 20 mpg using regular gas, how many mpg should the car get on premium gas to be equally cost effective?

2. How many integers between 1 and 2022 inclusive are multiples of 11 but not multiples of 3?

3. Suppose there are two concentric circles, the inner circle is inscribed in a square (the inner circle touches each side of the square) and the outer circle circumscribes the same square (the outer circle touches each vertex of the square). How many times larger is the area of the outer circle than the area of the inner circle?

4. In a neighborhood where the streets are arranged in a rectangular grid, a school lies 3 blocks north and 2 blocks east from a student's home. The student takes a 2-block detour to get more exercise. How many different 7-block paths can this student take to walk from home to school?

5. An election ballot has 4 issues on which a voter may mark yes, no, or abstain. If a voter is allowed to abstain on at most 2 issues, how many different ways are possible for a voter to mark the ballot?

6. How many different arrangements of the letters x and y are there of length at least 1 and not more than 8? (For example, xyx and xxy are two different arrangements, each of length 3.)

7. A large jar must be filled with a total of 2022 marbles, each of which is either red, blue, or green. At least one marble of each color must be used. For example, one way to fill the jar would be to use 600 red marbles, 400 blue marbles, and 1022 green marbles. Find N if $2021N$ is the number of different ways to fill the jar.

8. If you expand $(w+x+y+z)^{10}$, what is the coefficient of wx^2y^7?

9. Suppose there are 10 socks, 5 of which are black, and 5 of which are brown. You pick two socks out at a time (sampling without replacement), and each set of two forms a pair. So you just form 5 pairs of socks at random without worrying about matching. Of the five paired socks, let a/b be the probability that there are exactly 2 pairs that match, where the fraction is expressed in lowest terms. What is the numerator of this fraction?

10. Consider a standard deck of 52 cards. Four of the cards are aces, 13 of the cards are spades, and one of the cards is the ace of spades (i.e., both an ace and a spade). If the deck is shuffled, and two cards are drawn at random, let a/b be the probability that one card is an ace and the other is a spade, where the fraction is expressed in the lowest terms. What is the value of b?

Solutions

1. The answer is 21.

   For regular gas, the cost per mile is (4 \$/gal)/(20 mi/gal) = 1/5 \$/mi. To be equally cost effective using premium gas, if the car gets x mpg we need $\frac{4.20}{x} = \frac{1}{5}$, hence $x = (5)(4.20) = 21$.

2. The answer is 122.

   $2022/11 = 183$ plus a remainder, so there are 183 multiples of 11 between 1 and 2022.
   $2022/33 = 61$ plus a remainder, so there are 61 multiples of 3 and 11 between 1 and 2022.
   Thus the number of integers between 1 and 2022 that are multiples of 11 but not 3 are: $183 - 61 = 122$.

3. The answer is 2.

   Let r be the radius of the inner circle and R be the radius of the outer circle. Then the square must have side length $2r$ and diagonal $2R$. The Pythagorean theorem yields $2r^2 = R^2$, so the ratio of the circle areas is $\frac{\pi R^2}{\pi r^2} = 2$.

4. The answer is 245.

   Let N, E, W, S denote walking 1 block north, east, west, or south, respectively. A direct route with no detours would have 3 N's and 2 E's. There are 2 possible types of detours for 7-block paths, Type 1: an extra E and W; and Type 2: an extra N and S.
   The number of Type 1 paths is the number of arrangements of $3N$'s, $3E$'s, $1W$, which is

   $$C(7,3)C(4,3) = \frac{7!}{3!4!}\frac{4!}{3!1!} = \frac{7!}{3!3!} = \frac{7\cdot 6\cdot 5\cdot 4}{3\cdot 2\cdot 1} = 7\cdot 2\cdot 5\cdot 2 = 140.$$

   The number of Type 2 paths is the number of arrangements of $4N$'s, $2E$'s, $1S$, which is

   $$C(7,4)C(3,2) = \frac{7!}{4!3!}\frac{3!}{2!1!} = \frac{7!}{4!2!} = \frac{7\cdot 6\cdot 5}{2\cdot 1} = 7\cdot 3\cdot 5 = 105.$$

   The total of Type 1 and Type 2 paths is the sum: $140 + 105 = 245$.

5. The answer is 72.

   The number of ways to mark the ballot without restrictions is 3^4. From this we must subtract the number of ways to abstain on more than 2 issues. There are $C(4,3)$ ways to choose 3 issues to abstain on, and 2 ways to vote on the 1 remaining issue. There is only 1 way to abstain on all 4 issues. So the answer is

   $$3^4 - (C(4,3)\cdot 2 + 1) = 81 - (\frac{4!}{3!1!}\cdot 2 + 1) = 81 - (8+1) = 72.$$

6. The answer is 510.

   There are 2 length 1 choices, 2^2 length 2 choices, 2^3 length 3 choices, ... , 2^8 length 8 choices. Summing these yields

   $$2+2^2+2^3+2^4+2^5+2^6+2^7+2^8 = 2(1+2+2^2+\cdots+2^7) = 2\frac{1-2^8}{1-2} = 2(2^8-1) = 2(256-1) = 2(255) = 510.$$

7. The answer is 1010.

   The number of ways to fill the jar is the same as the number of ordered triples (x, y, z) of positive integers that are solutions to $x + y + z = 2022$.

   Method 1:
   case $x = 1$: $y + z = 2021$ has 2020 solutions (corresponding to choices for y, since z is then completely determined).
   case $x = 2$: $y + z = 2020$ has 2019 solutions.
   case $x = 3$: $y + z = 2019$ has 2018 solutions.
   $\vdots$
   case $x = 2019$: $y + z = 3$ has 2 solutions (namely $(y, z) = (1, 2)$ and $(y, z) = (2, 1)$).
   case $x = 2020$: $y + z = 2$ has 1 solution (namely $(y, z) = (1, 1)$).
   So the number of solutions is

   $$1 + 2 + 3 + 4 + \cdots + 2020 = \frac{(2020)(2021)}{2} = (1010)(2021) = 2021N$$

   where $N = 1010$.

   Method 2:
   The number of solutions is the same as the number of ways to leave 2 s's remaining from the initial placement of $2021s$'s interlaced between 2022 u's: $usususus\cdots su$. The remaining 2 s's separate the u's into 3 batches, the number of u's in each batch represent the values of x, y, and z. The number of ways to do this is

   $$C(2021,2) = \frac{2021!}{2!2019!} = \frac{(2021)(2020)}{2} = (2021)(1010) = 2021N$$

   where $N = 1010$.

8. This is just the multinomial theorem, so the answer is

$$\frac{10!}{1!2!7!0!} = 360.$$

9. Think of lining up the 10 socks in random order in a line. We pair socks 1 and 2, then 3 and 4, etc. There are $\binom{10}{5}$ ways to choose which socks are brown versus black. This is the denominator. For the numerator, since we have two pairs that match, they have to be one black and own brown pair. (If there were two black pairs, there'd be lots of brown socks left over, so you'd end up with brown matches as well). An arrangement that would work is (using L = black, R = brown):

$$(LL)(RR)(LR)(LR)(LR)$$

Think of this as

$$XYNNN$$

where X represents a black match, Y represents and brown match, and N represents a non-matched pair. There are

$$\frac{5!}{1!1!3!} = 20$$

ways to arrange the letters X, Y, N, N, N. For each N you also have two choices of which comes first, L or R. Thus the numerator is

$$20 \times 2^3 = 160$$

Therefore the probability of two matches is

$$\frac{160}{252} = \frac{40}{63}$$

So the answer is 40.

Alternate Method:
Treat each sock as distinguishable. Label the black socks L_1, L_2, L_3, L_4, L_5, and label the brown socks R_1, R_2, R_3, R_4, R_5. The total number of ways of drawing 5 pairs, 2 socks at a time is:

$$C(10,2)C(8,2)C(6,2)C(4,2)C(2,2) = \frac{10!}{2!8!}\frac{8!}{2!6!}\frac{6!}{2!4!}\frac{4!}{2!2!}\frac{2!}{2!0!} = \frac{10!}{2^5} = 113400.$$

For exactly 2 matches there must be one black match and one brown match, for example:

$$(L_iL_j)(R_kR_l)(L_mR_n)(L_pR_q)(L_rR_s)$$

where $ijmpr$ and $klnqs$ are each permutations of 12345. Thus there are (5!)(5!) ways of choosing the subscripts. However, interchanging i and j yields the same outcome, and similarly for k and l. Thus there are really only $(5!)(5!)/2^2$ ways of choosing subscripts for different outcomes. Finally, for each choice of subscripts, there are $5!/(1!1!3!)$ ways of arranging the 5 pairs of 3 different types: LL, RR, and LR. Thus the total number of ways to have exactly four matches is

$$\frac{5!5!}{2^2}\frac{5!}{3!} = 72000.$$

So the probability of exactly 2 matches is

$$\frac{72000}{113400} = \frac{40}{63},$$

and the numerator is 40.

10. Here let A_i indicate that card i is an ace, and S_i indicate that card i is spade. Here are the favorable cases and their probabilities:

$$\begin{aligned}
P(A_1S_1A_2S_2^c) &= (1/52)(3/51) = 3/(51\cdot 52)\\
P(A_1S_1A_2^cS_2) &= (1/52)(12/51) = 12/(51\cdot 52)\\
P(A_1S_1^cA_2S_2) &= (3/52)(1/51) = 3/(51\cdot 52)\\
P(A_1S_1^cA_2^cS_2) &= (3/52)(12/51) = 36/(51\cdot 52)\\
P(A_1^cS_1A_2S_2) &= (12/52)(1/51) = 12/(51\cdot 52)\\
P(A_1^cS_1A_2S_2^c) &= (12/52)(3/51) = 36/(51\cdot 52)
\end{aligned}$$

$$\text{Total} = \frac{3+12+3+36+12+36}{51\cdot 52} = \frac{102}{51\cdot 52} = \frac{1}{26}$$

Alternate Method:
The number of ways to draw 2 cards from 52 is

$$C(52,2) = \frac{52!}{2!50!} = \frac{52\cdot 51}{2} = (26)(51).$$

For the two drawn cards there are two cases for a favorable outcome:
Case 1: one card is a non-spade ace, the other card is a spade: $(3)(13) = 39$ possible ways.
Case 2: one card is the ace of spades and the other card is another spade: $(1)(12) = 12$ possible ways.
$39 + 12 = 51$ ways possible for a favorable outcome. So the probability of a favorable outcome is

$$\frac{51}{(26)(51)} = \frac{1}{26} = \frac{1}{b}$$

where $b = 26$.