UNM-PNM Statewide High School Mathematics Contest
Round-2, 10 February 2024, 14:00-17:30

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1. Consider a sum of natural numbers such that each digit from 1 to 8 appears only once. For example, the numbers 81, 27, 4536 feature each digit from 1 to 8 only once and sum to $81 + 27 + 4536 = 4644$.

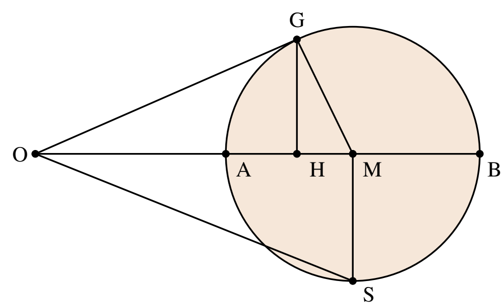
(a) Find such a sum which adds up to 243.

(b) Find the integer nearest 2024 which can be represented as such a sum.

2. In the figure points A, B, G, S are on a circle whose center is at M, and O is a point on the extension of the diameter AB. Moreover, OG is tangent to the circle, with both GH and SM perpendicular to the diameter AB. Suppose $a = \overline{OA}$ and $b = \overline{OB}$, with $0 < a < b$. Express the inequalities

$$\overline{OH} < \overline{OG} < \overline{OM} < \overline{OS}$$

in terms of a and b.



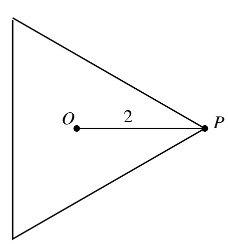
3. In the hexadecimal (base-16) number system a (nonstandard) notation for the base symbols is the following.

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, u, v, w, x, y, z$$

(a) Find the hexidecimal representation of the decimal number $(301.5)_{10}$.

(b) Find the decimal representation of the hexadecimal number $(z.\overline{u9v})_{16}$. You may give your answer as a fraction in simplest form.

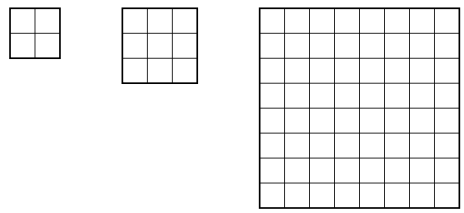
4. Let $\mathcal{T}$ be a solid equilateral triangle, with center point O. Suppose the shown segment with endpoints O and P has length 2. Define the region $\mathcal{R} \subset \mathcal{T}$ to be all points Q inside $\mathcal{T}$ such that the triangle ΔOPQ has an obtuse angle. What is the area of $\mathcal{R}$?



5. Consider the game boards shown in the figure, respectively with 4, 9, and 64 squares or *cells.*

(a) For the 4-cell board how many possible ways can 2 cells be chosen from the board? If 2 cells of the 4-cell board are chosen at random, then what is the probability they will have a common side?

(b) Answer the same questions for the 9-cell board and the 64-cell board.



6. Recall the absolute value function

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x \leq 0, \end{cases}$$

and consider an ordered pair (x, y) of real numbers. Viewing (x, y) as a two-component vector, its *weighted 1-norm* is

$$\big\|(x, y)\big\|_{(w_1, w_2)} = w_1|x| + w_2|y|,$$

where w_1 and w_2 are strictly positive real numbers call the *weights.*

(a) Sketch the region in the plane (in fact a polygon) whose points obey $\big\|(x, y)\big\|_{(4,2)} \leq 10$.

(b) The region determined by $\big\|(|x| + 1, |y| - 2)\big\|_{(4,2)} \leq 10$ is also a polygon. Specify it.

7. Let players A and B take turns flipping a fair coin with player A going first. The players flip the coin until a tails occurs after a heads. The first player to toss tails immediately after a heads wins. Find the probability that player A wins.

8. Given $p(x) = \frac{1}{2}x^2 - x + \frac{1}{3}$, consider the quadratic equation $p(x) = 0$. Starting with $x_0 = 0$, the iterative scheme

$$x_{k+1} = x_k - \frac{\frac{1}{2}x_k^2 - x_k + \frac{1}{3}}{x_k - 1},$$

generates a *Newton sequence* $x_0, x_1, x_2, \ldots$ which converges to a root of the quadratic equation.

(a) Find the roots x_- and x_+ of this quadratic equation, assuming $x_- < x_+$. Write down the first three terms x_0, x_1, x_2 of the Newton sequence.

(b) Assuming $0 \leq x_k < x_-$, show that (i) $x_k < x_{k+1}$ and (ii) $x_{k+1} < x_-$. *Hint: for (ii) consider* $p(x_k + (x_- - x_k))$.

(c) Specify the real numbers a and θ in the following exact formula for the iterates in the sequence.

$$x_k = a\Big(\frac{1 - \theta^{2^k - 1}}{1 - \theta^{2^k}}\Big)$$