UNM-PNM Statewide High School Mathematics Contest
Round-1 Contest, 11-13 November 2023

• Please do the following **ten** problems over a continuous **three-hour period**.

• The solution to each problem is either an integer n, an expression $n\pi$, a fraction p/q in **lowest terms**, or an expression $p\pi/q$. As your answer, enter either n or the denominator q.

Example problem	**Solution**	**Entered answer**
What is the area of a square of side length 3?	9	9
What is the area of a square of side length $\frac{1}{2}$?	$\frac{1}{4}$	4
What is the circumference of a circle of radius 1?	2π	2
What is the circumference of a circle of radius $\frac{3}{7}$?	$\frac{6}{7}\pi$	7

• Because grading will be automated, all answers should be entered as integers. For example, write `2`, not `2.0` or `two`. Do not include commas; write `100421`, not `100,421`.

• Express angles in radians, not degrees.

• No calculators permitted.

Student (LAST name, FIRST name): ______________________________

Student email address: ______________________________

Grade level: ______________________________

School or club: ______________________________

Teacher or coach: ______________________________

1. A hole is drilled in a sheet of aluminum. For 20 seconds the steel drill bit spins at 50 revolutions per second, and then for 10 seconds it spins at 80 revolutions per second until the hole is opened. For the purposes of the problem, you may view the transition from 50 to 80 revolutions per second as occurring instantaneously. What is the drill bit's average rate of rotation expressed in revolutions per second?

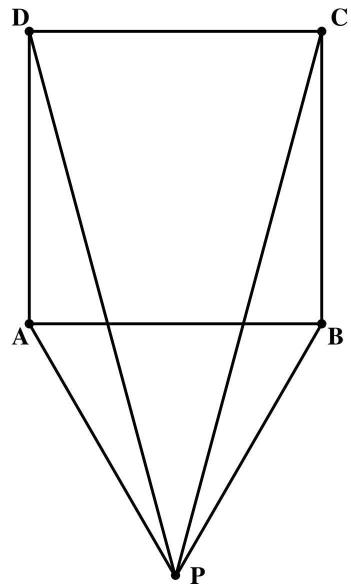
ENTERED ANSWER:
60

Solution. The total number of revolutions is

$$(20\,\texttt{sec})(50\,\texttt{rev/sec}) + (10\,\texttt{sec})(80\,\texttt{rev/sec}) = 1000\,\texttt{rev} + 800\,\texttt{rev} = 1800\,\texttt{rev},$$

achieved in a total of $30\,\texttt{sec}$. The average rotation rate is therefore $1800\,\texttt{rev}/(30\,\texttt{sec}) = 60\,\texttt{rev/sec}$.

2. Suppose the point P outside of the square $ABCD$ is such that ΔABP is an equilateral triangle. What is the angle $\angle CPD$?



ENTERED ANSWER:
6 (answer 30° also accepted)

Solution. The triangles ΔADP and ΔBCP are isosceles. Since $\angle DAP = \angle CBP = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$, the remaining angles (for example $\angle BPC$) are $\frac{\pi}{12}$. Therefore, $\angle DPC = \frac{\pi}{3} - 2\frac{\pi}{12} = \frac{\pi}{6}$.

3. Find the largest value of x such that $|x-50|-|x-112| \leq 0$.

ENTERED ANSWER:
81

Solution. Thinking geometrically, to have $|x-50|-|x-112|$ negative, we need a point x on the number line that is closer to 50 than to 112. The number 81 is equidistant between the two, so the solution to the inequality is $\{x : x \leq 81\}$, and therefore $x = 81$ is the largest value satisfying the inequality.

In more detail, notice that for $x \geq 112$ we can write

$$|x-50|-|x-112| = x-50-(x-112) = 62,$$

and similarly for $x \leq 50$

$$|x-50|-|x-112| = -(x-50)+(x-112) = -62.$$

For $50 < x < 112$

$$|x-50|-|x-112| = x-50+(x-112) = 2x-162.$$

The last expression is clearly nonpositive for $x \leq 81$. Therefore, the inequality is satisfied for $x \in (-\infty, 81]$. Again, $x = 81$ is the largest value of x for which the inequality holds true.

4. In double precision computer arithmetic, a real number occupies 64 `bits` or 8 `bytes` (a `bit` is either a `1` or a `0`, and a `byte` is an 8-`bit` string). Suppose that a desktop computer has 8 `GB` (8×10^9 `bytes`) of random-access memory. What is the largest integer value n such that eight two-dimensional arrays, each of size $n \times n$, can fit in the computer memory? *Hint:* $\sqrt{5} \simeq$ `2.23606797749979`.

ENTERED ANSWER:
11180 (answers 31622 or 89442 also accepted)

Solution. 8 `GB` corresponds to 8×10^9 bytes, which corresponds to storage of

$$\frac{8 \times 10^9\,\texttt{bytes}}{8\,\texttt{bytes/double}} = 10^9\,\texttt{doubles},$$

where `double` stands for a double precision real number. Therefore, each of the eight arrays holds at most $0.125 \times 10^9\,\texttt{doubles} = 125 \times 10^6\,\texttt{doubles}$. Therefore, the maximum n is

$$n = \lfloor\sqrt{125 \times 10^6}\rfloor = \lfloor 5000 \cdot \sqrt{5}\rfloor = \lfloor 5 \cdot 2236.06797749979\rfloor = 11180.$$

Note that the hint has been used.

Remark. The problem meant the arrays were of double precision numbers (8 `bytes`). Conceivably, in the problem reading the arrays might be interpreted as holding numbers which require either `byte` or `bit` storage. This was not the intent, but let us consider these two interpretations.

If the arrays are of numbers which require only the storage of a `byte` (not a common type in computer languages), then 8 `GB` corresponds to storage of

$$\frac{8 \times 10^9\,\texttt{bytes}}{1\,\texttt{byte/number}} = 8 \times 10^9\,\texttt{numbers}.$$

Therefore, each of the eight arrays holds at most $10^9\,\texttt{numbers}$. In this case the maximum n is

$$n = \lfloor\sqrt{10^9}\rfloor = \lfloor\sqrt{10} \cdot 10000\rfloor = \lfloor 3.16227766016838 \cdot 10000\rfloor = 31622.$$

If the arrays hold numbers which are only `bits`, then 8 `GB` corresponds to storage of

$$64 \times 10^9\,\texttt{numbers}.$$

Then each of the eight arrays holds at most $8 \times 10^9\,\texttt{numbers}$. In this final case the maximum n is

$$n = \lfloor\sqrt{8 \times 10^9}\rfloor = \lfloor 2\sqrt{20} \cdot 10000\rfloor = 89442.$$

5. In the hexadecimal (base-16) number system a (nonstandard) notation for the base symbols is the following.

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, u, v, w, x, y, z$$

Find the decimal (base-10) representation of the following hexadecimal representation: $(y420.0)_{16}$.

ENTERED ANSWER:
58400

Solution. The expansion is

$$\begin{aligned}(y420.0)_{16} &= 14 \cdot 16^3 + 4 \cdot 16^2 + 2 \cdot 16^1 + 0 \cdot 16^0 \\ &= 16(14 \cdot 16^2 + 4 \cdot 16 + 2) \\ &= 16(14 \cdot 2^8 + 4 \cdot 2^4 + 2) \\ &= 32(14 \cdot 2^7 + 4 \cdot 2^3 + 1) \\ &= 32(14 \cdot 128 + 4 \cdot 8 + 1) \\ &= 32(1792 + 32 + 1) \\ &= 32(1825) \\ &= 58400.\end{aligned}$$

6. Let a bowl have five marbles labeled $1, 2, 3, 4, 5$. Suppose two marbles are drawn randomly without replacement from the bowl. Compute the probability that both marbles have an even number given that at least one of them does.

ENTERED ANSWER: 7

Solution. One approach is to write out the sample space as

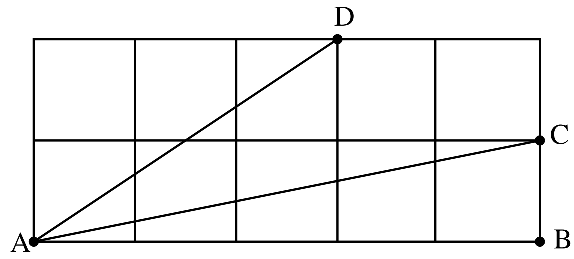
$$\begin{aligned}
&(1,2),(1,3),(1,4),(1,5),\\
&(2,1),(2,3),(2,4),(2,5),\\
&(3,1),(3,2),(3,4),(3,5),\\
&(4,1),(4,2),(4,3),(4,5),\\
&(5,1),(5,2),(5,3),(5,4).
\end{aligned}$$

These are possible draws of two marbles. There are 14 draws in the sample space where at least one number is even:

$$\begin{aligned}
&(1,2),(1,4),\\
&(2,1),(2,3),(2,4),(2,5),\\
&(3,2),(3,4),\\
&(4,1),(4,2),(4,3),(4,5),\\
&(5,2),(5,4).
\end{aligned}$$

For two of these both numbers are even: (2,4) and (4,2). Therefore, the probability is $\frac{2}{14}$ or $\frac{1}{7}$.

7. The rectangle shown is partitioned into 10 congruent squares, each 1×1 in size. Find the sum of the two angles $\angle BAC$ and $\angle BAD$.

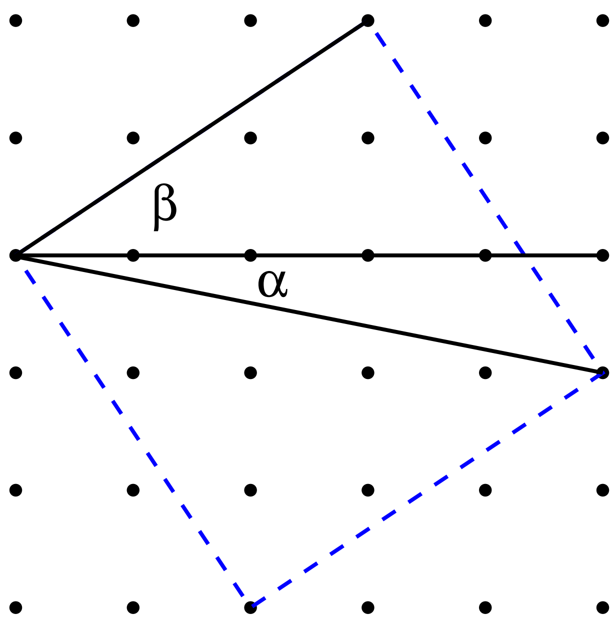


ENTERED ANSWER: 4 (answer 45° also accepted)

Solution. The two angles have respective tangents $\tan\alpha = \frac{1}{5}$ and $\tan\beta = \frac{2}{3}$. Then

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta} = \frac{\frac{1}{5} + \frac{2}{3}}{1 - \frac{1}{5}\cdot\frac{2}{3}} = 1.$$

The angle is then $\frac{1}{4}\pi$. Here is a graphical solution.



8. How many integers between 1 and 2023 are multiples of 5, 7, or 17?

ANSWER: 718 (answer 717 also accepted)

Solution. Let A_i be the set of integers between 1 and 2023 that are multiples of i. Thus, A_5 is the set of integers between 1 and 2023 divisible by 5, and $|A_5| = \lfloor 2023/5 \rfloor = 404$, $|A_7| = 2023/7 = 289$, and $|A_{17}| = 119$.

The set of integers that are multiples of 5, 7, or 17 can be found by inclusion-exclusion. Let $A_{5,7}$, for example, be the set of integers between 1 and 2023 divisible by both 5 and 7; this is the set of integers between 1 and 2023 divisible by 35. Then

$$|A_5 \cup A_7 \cup A_{17}| = |A_5| + |A_7| + |A_{17}| - |A_{5,7}| - |A_{5,17}| - |A_{7,17}| + |A_{5,7,17}|$$

We have $|A_{5,7}| = |A_{35}| = \lfloor 2023/35 \rfloor = 57$, $|A_{5,17}| = |A_{85}| = 23$, and $|A_{7,17}| = |A_{119}| = 17$, and $|A_{5,7,17}| = |A_{593}| = 3$. Therefore,

$$|A_5 \cup A_7 \cup A_{17}| = 404 + 289 + 119 - 57 - 23 - 17 + 3 = 718.$$

2023 is divisible by 7 and 17, but not 5. If "between 1 and 2023" is read as not inclusive of the endpoints, then the answer is 717.

9. Let x_1, x_2, x_3 denote the (possibly complex) roots of the cubic equation

$$2x^3 + 12x^2 + 30x + 30 = 0.$$

Evaluate the the expression

$$x_1x_2 + x_1x_3 + x_2x_3.$$

ANSWER: 15

Solution. The monic cubic polynomial $p(x) = x^3 + 6x^2 + 15x + 15$ has the same roots as the polynomial appearing in the given equation. In factored form

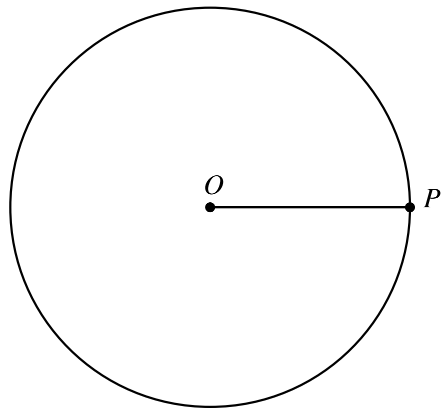
$$p(x) = (x - x_1)(x - x_2)(x - x_3) = x^3 - x^2(x_1 + x_2 + x_3) + x(x_1x_2 + x_1x_3 + x_2x_3) - x_1x_2x_3,$$

and from this equation

$$x_1 + x_2 + x_3 = -6, \quad x_1x_2 + x_1x_3 + x_2x_3 = 15, \qquad x_1x_2x_3 = -15.$$

Therefore, the given expression has value 15.

10. Let C be a circle of radius 2 centered at the point O, with P a point on the circle. Define the region R to be all points Q inside the circle such that the triangle ΔOPQ has an obtuse angle (one which exceeds $\frac{1}{2}\pi$). What is the area of R?



ENTERED ANSWER:
3

Solution. For points within the smaller circle shown, $\angle OQP$ is obtuse, whereas if Q where on the smaller circle $\angle OQP = \frac{1}{2}\pi$. For points in the shown left-half disk $\angle POQ$ is obtuse. Since C has radius $r = 2$, the area of the combined shaded regions is $\pi(r/2)^2 + \frac{1}{2}\pi r^2 = 3\pi$.